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عنوان الرسالة: Some properties of the ring of continuous functions vanishing at infinity

اعلن بأنني قد التزمت بقوانين الجامعة الأردنية وأنظمتها وتعليماتها وقراراتها السارية المفعول المتعلقة باعداد رسائل الماجستير عندما قمت شخصيا" باعداد رسالتي وذلك بما ينسجم مع الأمانة العلمية وكافة المعايير الأخلاقية المتعارف عليها في كتابة الرسائل العلمية. كما أنني أعلن بأن رسالتي هذه غير منقولة أو مستلة من رسائل أو كتب أو أبحاث أو أي منشورات علمية تم نشرها أو تخزينها في أي وسيلة اعلامية، وتأسيسا" على ما تقدم فانني أتحمل المسؤولية بأنواعها كافة فيما لو تبين غير ذلك بما فيه حق مجلس العمداء في الجامعة الأردنية بالغاء قرار منحي الدرجة العلمية التي حصلت عليها وسحب شهادة التخرج مني بعد صدورها دون أن يكون لي أي حق في التظلم أو الاعتراض أو الطعن بأي صورة كانت في القرار الصادر عن مجلس العمداء بهذا الصدد.

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SOME PROPERTIES OF THE RING OF CONTINUOUS FUNCTIONS VANISHING AT INFINITY

By

Amani Khalil Shatara

Supervisor

Dr. Emad Abu Osba

This Thesis was Submitted in Partial Fulfillment of the Requirements for the Master's Degree in Science in Mathematics.

Faculty of Graduate Studies
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April, 2011

The University of Jordan

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Dedication

To my memory of my mother

To my father

To my sisters & brothers

To my supervisor Dr. Emad Abu Osbai

To Dr. Mufid Azzam

To committee Debate

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SOME PROPERTIES OF THE RING OF CONTINUOUS FUNCTIONS VANISHING AT INFINITY

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ABSTRACT

Let X be a completely regular Hausdorff space, and let C(X) be the ring of all continuous real valued functions defined on X. Let $C_\infty(X)$ be the set of all functions vanishing at infinity, i.e. the functions f in C(X) such that $\{x \in X : |f(x)| \geq \frac{1}{n}\}$ is compact for all $n \in \mathbb{N}$.

In this thesis, we study the subring $C_\infty(X)$. We show that it is a subring of C(X), but need not be an ideal. We show that $C_\infty(X)$ is an ideal in C(X) if and only if **e**very open locally compact subset of *X* is bounded if and only if every open locally compact σ-compact subset of *X* is bounded. We also show that if *X* is a locally compact Hausdorff space, then $C_\infty(X)$ is an ideal in C(*X*) if and only if *X* is pseudocompact space.

We also characterize prime and maximal ideals in $C_\infty(X)$. We show that the prime ideals of $C_\infty(X)$ are $\{ P^* \cap C_\infty(X) : P^* is$ a prime ideal in $C^*(X)$ and $C_\infty(X) \not\subset P^*\}$.

We show that the set of maximal ideals in $C_\infty(X)$ is $\{M_x \cap C_\infty(X)$: x has a compact neighborhood$\}$. We also show that $C_\infty(X)$ may contains a proper ideal which is not contained in any maximal ideal of $C_\infty(X)$.

Introduction

Let X be a completely regular Hausdorff space, and let C(X) be the ring of all continuous real valued functions defined on X. A lot of material was established on this ring, see for example (Gillman and Jerison, 1976).

Let $C_\infty(X)$ be the set of all functions vanishing at infinity, i.e. the functions *f* such that $\{x \in X : |f(x)| \geq \frac{1}{n}\}$ is compact for all $n \in \mathbb{N}$. This subring was defined in (Gillman and Jerison, 1976) and some of its basic properties were assigned as exercises. Nowadays a lot has been achieved on this subring, especially the case at which it becomes an ideal, see for example (Azarpanah and Soundarar, 2001).

In this thesis, we study the subring $C_\infty(X)$. The material in this thesis is given in four chapters.

In chapter 1, we will review some basic facts from algebra and topology which will be used later on in this thesis. We will also introduce the ring C(X) and some of its basic properties.

In chapter 2, we will define the subring $C_\infty(X)$ and investigate its general properties. We also investigate the relations between this subring and the ideal $C_K(X)$, the ideal of functions in C(X) with compact support.

In chapter 3, we will study when $C_\infty(X)$ is an ideal in $C(X)$ and investigate the relation between this ideal and some ideals in $C(X)$.

In chapter 4, we will study the structure of the prime and maximal ideals in $C_\infty(X)$, we will also show that some proper ideals in $C_\infty(X)$ may not be contained in a maximal ideal. We will study essential and uniform ideals in $C_\infty(X)$ too.

In this thesis, we gathered the material from different references and wrote all necessary details, we also gave examples and counter examples when it is necessary. We also found $C_\infty(X)$ for some famous spaces X.

Chapter 1

Preliminaries

In this chapter, we will review some basic facts from algebra and topology which will be used later on in this thesis. We will also introduce the ring C(X) and some of its basic properties.

All rings in this thesis are assumed to be commutative, and all topological spaces are assumed to be completely regular T_2-spaces .

The material of this chapter can be found in (Willard, 1970), (Gillman and Jerison, 1976), (Hungerford, 1996), and (Abu Osba, 1992).

In this chapter, we gather the material from different resources, write them all together, and supply some examples and counter examples when necessary.

1.1 Basics of ring theory

Definition 1.1.1 Let R be a ring.

(a) A non empty subset I of a ring R is called an ideal if:

(i) For every $a,b \in I$, $a-b \in I$.

(ii) For every $a \in I$ and $r \in R$, $ar \in R$.

(b) An element $a \in R$ is called idempotent if $a^2 = a$.

(c) An ideal $P \neq R$ is called prime if whenever *a* and *b* are elements of R with $ab \in P$, then $a \in P$ *or* $b \in P$.

(d) For any $a \in R$, an ideal (a) generated by a single element *a* is called a principal ideal of the ring R.

(e) A proper ideal *M* of *R* is called maximal ideal if whenever $M \subseteq N \subseteq R$ where *N* is an ideal in *R*, then $M = N$ *or* $N = R$.

Example 1.1.2 In $\mathbb{Z}$, the ideal (2) is both maximal and prime ideal, but (4) is neither maximal nor prime.

Theorem 1.1.3 In a non-zero ring R with unity, maximal ideals always exist.

In fact every ideal except R is contained in a maximal ideal.

Theorem 1.1.4 If I and J are ideals in R neither containing the other, then I $\cap$ J is not prime.

Proof: Let $f \in$ I – J, $g \in$ J – I. Then $fg \in$ I $\cap$ J, but neither f nor g belongs to I $\cap$ J. Therefore I $\cap$ J is not prime. #

Definition 1.1.5 Let R be a ring. Then for any $a \in R$, $Ann(a) = \{r \in R : ar = 0\}$.

Definition 1.1.6 For any ring R, $Max(R) = \{M : M$ is a maximal ideal in R$\}$.

Definition 1.1.7 Let R and S be rings. A function $f : R \longrightarrow S$ is a homomorphism of rings if for all $a, b \in R$:

(i) $f(a+b) = f(a) + f(b)$.

(ii) $f(ab) = f(a)f(b)$.

If f is an onto homomorphism, then it is called an epimorphism.

The set $Ker(f) = \{r \in R : f(r) = 0\}$, is called the kernel of f.

Theorem 1.1.8 Let M be a proper ideal in a ring R. Then

(1) M is maximal ideal if and only if R/M is field.

(2) M is a prime ideal if and only if R/M is an integral domain.

Theorem 1.1.9 In a ring R, each maximal ideal is prime, but the converse need not be true.

For example, in $\mathbb{Z}[x]$ the ideal (2) is prime but not maximal.

Definition 1.1.10 Let R be a ring. Then R is said to be regular if for each $a \in R$ there exists an element $b \in R$ such that $a = a^2 b$.

Example 1.1.11 The ring $\mathbb{Z}_6$ is a regular ring but $\mathbb{Z}_4$ is not, since in $\mathbb{Z}_4$, $\overline{2}^2 = \overline{0}$ so $\overline{2} \neq \overline{2}^2 . x$ for all $x \in \mathbb{Z}_4$.

Theorem 1.1.12 A ring R is called a zero dimensional ring if every prime ideal in R is maximal.

Definition 1.1.13 Let R be a ring. Then R is called a reduced ring if whenever $a \in$ R and $a^n = 0$, for some n $\in \mathbb{N}$, then $a = 0$.

Definition 1.1.14 Let R be a commutative ring with unity. An R-algebra is a ring A with unity together with a ring homomorphism $f : R \longrightarrow A$ mapping 1_R to 1_A such that the subring $f(R)$ of A is contained in the center of A.

1.2 Basics of Topology

Theorem 1.2.1 Let U be an open set in a topological space X. For any subset A of X, $Cl_X(U \cap A) = Cl_X(U \cap Cl_X A)$.

Proof: Since $U \cap A \subseteq U \cap Cl_X A$, it follows that $Cl_X(U \cap A) \subseteq Cl_X(U \cap Cl_X A)$.

Now, Let $x \in Cl_X(U \cap Cl_X A)$ and $x \notin Cl_X(U \cap A)$. Then there exists an open set V containing x such that $V \cap U \cap A = \varnothing$. i.e. $(V \cap U) \cap A = \varnothing$. Hence $A \subseteq X \setminus (V \cap U)$, which implies that $Cl_X A \subseteq X \setminus (V \cap U)$.

Thus $(V \cap U) \cap Cl_X A = V \cap (U \cap Cl_X A) = \varnothing$, which is a contradiction. Therefore $Cl_X(U \cap A) = Cl_X(U \cap Cl_X A)$. #

Theorem 1.2.2

a) Every closed subset of a compact space is compact.

b) A compact subset of a Hausdroff space is closed.

c) The continuous image of a compact space is compact.

Theorem 1.2.3 A compact $T_2 - space$ is normal, and therefore it is completely regular.

Definition 1.2.4 A space *X* is locally compact if each point in *X* has a neighborhood base consisting of compact sets.

Theorem 1.2.5 A $T_2 - space$ *X* is locally compact if and only if for each point *x* in *X* there exists an open set *U* containing *x* such that $Cl_X U$ is compact.

Example 1.2.6 The real line $\mathbb{R}$ is locally compact, but the set of rationals $\mathbb{Q}$ is not locally compact.

Theorem 1.2.7 Every locally compact Hausdroff space is a Tychonoff.

Definition 1.2.8 A space *X* is Lindelöff if every open cover of *X* has a countable subcover.

Definition 1.2.9 A space X is $\sigma-compact$ if it can be written as a union of countably many compact subsets.

Theorem 1.2.10 Every $\sigma-compact$ space is Lindelöff.

Proof: Assume $\underset{\sim}{U} = \{ U_\alpha : \alpha \in \Delta\}$ is an open cover of X.

Let $X = \bigcup_{i=1}^{\infty} A_i$, with A_i is compact for all i. Then $A_i \subseteq \bigcup_{i=1}^{n} U_{\alpha_i}$, hence $X \subseteq \bigcup_{i=1}^{\infty}\bigcup_{j=1}^{n} U_{\alpha_j}$.

Thus $\underset{\sim}{U}$ has a countable subcover and so X is Lindelöff. #

Theorem 1.2.11 A space X is countably compact if and only if every infinite countable set has a cluster point.

1.3 The ring C(X)

Let X be a topological space. Let C(X) denote the set of all continuous real valued functions defined on the space X. C(X) is provided with a ring structure through addition and multiplication defined by:

$(f+g)(x)=f(x)+g(x)$ and $(f.g)(x)=f(x).g(x)$ for any $f, g \in C(X)$.

It is obvious that both operations are associative, commutative, and the distributive laws hold.

Hence C(X) is a ring with the zero element being the constant function 0, and the unity being the constant function 1.

The additive inverse $-f$ of f is characterized by the function $(-f)(x)=-f(x)$. The multiplicative inverse f^{-1} (in case it exists) is given by the formula

$$f^{-1}(x)=\frac{1}{f(x)}.$$

Example 1.3.1 The set of all continuous real valued functions, which are defined on the interval [0, 1] is a ring with respect to the operations that are defined above. This ring is denoted by C ([0, 1]).

Definition 1.3.2 Let $C^*(X)$ denote the set of all bounded functions in C(X).

Note that if $f, g \in C^*(X)$, then $f-g$, $f.g \in C^*(X)$. Thus $C^*(X)$ is a subring of C(X).

Theorem 1.3.3 Let X be a topological space and $x \in X$. Then

$M_x = \{f \in C(X): f(x) = 0\}$ is a maximal ideal of C(X).

Proof: Let $x \in X$. Then define the mapping

$\psi : C(X) \longrightarrow \mathbb{R}$ by $\psi(f) = f(x)$, where $f \in C(X)$. It is clear that ψ is a ring epimorphism. Hence by the first isomorphism theorem, we have

$C(X) / \ker(\psi) \cong \mathbb{R}$, but

$\ker(\psi) = \{f \in C(X): \psi(f) = 0\} = \{f \in C(X): f(x) = 0\} = M_x$.

Thus $C(X) / M_x \cong \mathbb{R}$ which is a field, and so M_x must be a maximal ideal in C(*X*).

Definition 1.3.4 A subset *S* of a space *X* is said to be a zero set if there exists

$f \in C(X)$ such that $S = f^{-1}(0)$. In this case we write $S = Z(f)$. It is clear that for any

$f \in C(X)$, $Z(f)$ is a closed set in X, since $\{0\}$ is closed in $\mathbb{R}$.

Thus we have $Z(f) = \{x \in X : f(x) = 0\}$, and let

$coz\ f = X - Z(f) = \{x \in X : f(x) \neq 0\}$,

The support of f is defined as supp $f = \overline{coz\ f}$. For each ideal I of C(X),

$coz\ I = \bigcup_{f \in I} coz\ f$, and $Z[I] = \{Z(f) : f \in I\}$.

It is clear that supp f is closed in X, while coz f is open in X.

In the following, we will give some important properties of the zero sets.

Theorem 1.3.5 The ring C(X) is a reduced ring.

Proof: Let $f \in$ C(X), and suppose that $f^n = 0$, for some $n \in \mathbb{N}$.

Then $Z(f^n) = X$. Since $Z(f^n) = Z(f)$, it follows that $Z(f) = X$.

Hence $f = 0$. #

Lemma 1.3.6 Let X be a topological space and $f, g \in C(X)$. Then:

(i) $Z(fg) = Z(f) \cup Z(g)$.

(ii) $Z(f^2 + g^2) = Z(f) \cap Z(g) \subseteq Z(f - g)$.

(iii) f is a unit in C(*X*) if and only if $Z(f) = \varnothing$.

Proof: (i) Let $x \in Z(fg)$. Then $fg(x) = 0$, and hence $f(x) = 0$ *or* $g(x) = 0$.

Therefore $x \in Z(f) \cup Z(g)$, and so $Z(fg) \subseteq Z(f) \cup Z(g)$.

Now, Let $x \in Z(f) \cup Z(g)$. Then $f(x) = 0$ *or* $g(x) = 0$.

Thus $fg(x) = 0$, hence $x \in Z(fg)$ and $Z(f) \cup Z(g) \subseteq Z(fg)$.

Therefore $Z(fg) = Z(f) \cup Z(g)$.

(ii) The element $x \in Z(f^2 + g^2)$ if and only if $f^2(x) + g^2(x) = 0$ if and only if

$f(x) = 0$ and $g(x) = 0$ if and only if $x \in Z(f) \cap Z(g)$. So,

$Z(f^2 + g^2) = Z(f) \cap Z(g)$.

Now, let $x \in Z(f) \cap Z(g)$. Then $f(x) = 0$ and $g(x) = 0$. So $f(x) - g(x) = 0$.

Therefore $x \in Z(f - g)$, and so $Z(f) \cap Z(g) \subseteq Z(f - g)$.

(iii) Let f be a unit. Then there exists $g \in C(X)$, such that $fg = 1$.

Hence for every $x \in X$, $f(x) \neq 0$. Therefore, $Z(f) = \varnothing$.

Conversely, let $Z(f) = \varnothing$. Then $g(x) = \dfrac{1}{f(x)} \in C(X)$. Hence $fg = 1$.

Therefore f is a unit. #

Definition 1.3.7 An ideal *I* of C(*X*) is called a z-ideal provided that if $Z(f) \in Z[I]$,

then $f \in I$. It is called a fixed ideal if $\bigcap_{f \in I} Z(f) \neq \varnothing$, otherwise it is called free.

Example 1.3.8 If M is a maximal ideal in C(X), then M is a z-ideal.

Theorem 1.3.9 For any proper z-ideal I in C(X), I is prime ideal if and only if I contains a prime ideal.

Theorem 1.3.10 The intersection of arbitrary non-empty family of z-ideals is a z-ideal.

Proof: Suppose that $\{I_\lambda : \lambda \in \Lambda\}$ is a family of z-ideals.

Let $Z(f) \in Z[\bigcap_{\lambda \in \Lambda} I_\lambda]$. Then $Z(f) \in Z[I_\lambda]$ for all $\lambda \in \Lambda$, but I_λ is a z-ideal, so it

follows that $f \in I_\lambda$ for all $\lambda \in \Lambda$. Therefore $f \in \bigcap_{\lambda \in \Lambda} I_\lambda$.#

Definition 1.3.11 Assume *X* is a topological space and *M* is a subspace of *X*.

Let $i_M : M \longrightarrow X$ such that $i_M(x) = x$. Then for each *U* open in *X*,

$i_M^{-1}(U) = U \cap M$. The function i_M is called an embedding function of *M* into *X*.

Definition 1.3.12 The subspace S of X is C^*-embedded in X if every function in $C^*(S)$ can be extended to a function in $C^*(X)$.

Definition 1.3.13 Two subsets A and B of a space X are said to be completely separated if there exists a function f in C(X) such that

$0 \leq f \leq 1$, $f(x) = 0$, for all $x \in A$ and $f(x) = 1$, for all $x \in B$.

Theorem 1.3.14 Two sets are completely separated if they are contained in disjoint zero sets. Moreover, completely separated sets have disjoint zero sets neighborhood.

Theorem1.3.15 A subspace S of X is C^*- embedded in X if and only if any two completely separated sets in S are completely separated in X.

Note that every closed set in a metric space is C^*- embedded.

Definition 1.3.16 A compactification of a space X is an ordered pair (K, h) where K is a compact $T_2-space$ and h is an embedding of X as a dense subset of K.

Theorem 1.3.17 Every (completely regular) space X has a compactification βX with the following equivalent properties:

(i) Every continuous mapping τ from X into any compact space Y has a continuous extension τ^{β} from βX into Y.

(ii) Every function $f \in C^*(X)$ has an extension to a function $f^{\beta} \in C(\beta X)$.

(iii) Any two disjoint zero sets in X have disjoint closures in βX .

(iv) For any two zero sets Z_1, Z_2 in X, $Cl_{\beta X}(Z_1 \cap Z_2) = Cl_{\beta X} Z_1 \cap Cl_{\beta X} Z_2$.

For any topological space X, βX is called the Stone-Čech compactification of X, in which X is dense. Moreover, βX is homeomorphic to the maximal ideal space of the ring $C(X)$, $Max(C(X))$. For more details about βX, see (Gillman and Jerison, 1976).

Definition 1.3.18 For each $p\in \beta X$, let $O^P = \{f \in C(X): p \in Int_{\beta X} Cl_{\beta X} Z(f)\}$,

$O^K = \bigcap_{p\in K} O^p$, for $K\subseteq \beta X$. Also let $M^p = \{f \in C(X): p \in Cl_{\beta X} Z(f)\}$ and

$M^K = \bigcap_{p\in K} M^p$, for $K\subseteq \beta X$.

Lemma 1.3.19 Let $f \in C(X)$. Then $f \in O^x$ if and only if there exists an open set *V* in βX containing *x* such that $V \bigcap X \subseteq Z(f)$.

Proof: If $f \in O^x$, then there exists an open set *V* in βX such that

$x \in V \subseteq Cl_{\beta X} Z(f)$. So $\emptyset \neq V \cap X \subseteq Z(f)$, since *X* is dense in βX.

Conversely, if $V \bigcap X \subseteq Z(f)$, where *V* is an open set in βX containing *x*, then

$V \subseteq Cl_{\beta X} V = Cl_{\beta X}(V \cap \beta X) = Cl_{\beta X}(V \cap X) \subseteq Cl_{\beta X} Z(f)$.

Hence $x \in V \subseteq Cl_{\beta X} Z(f)$. Therefore $f \in O^x$. #

Theorem 1.3.20 For every $p\in X$, $O^p = O_p = \{f \in C(X): p \in Int_X Z(f)\}$.

Proof: Let $f \in O^p$. Then there is an open set *V* in βX containing *p* such that

$V \bigcap X \subseteq Z(f)$.

Hence $f \in O_p$ since $V \cap X$ is an open set in *X* containing *p*.

Conversely, if $f \in O_p$, then there exists an open set *U* in *X* such that

$x\in U \subseteq Z(f)$. Thus there exists an open set *V* in βX such that $U = V \cap X$.

Therefore $V \cap X \subseteq Z(f)$, and by 1.3.18, we get that $f \in O^p$. #

Theorem 1.3.21 Let *M* be a nbhd of $y\in X$. Then there exists $g \in O_y$

such that $Z(g) \subseteq M$.

Proof: Let *M* be a nbhd of $y \in X$. Then there exists an open set *U* such that

$y \in U \subseteq M$.

Thus $X \setminus U$ is closed set which does not contain y. But *X* is completely regular, hence

there exists $f \in C(X)$ such that $f(y) = 0$ and $f(X \setminus U) = 1$.

Let $g(x) = (f(x) - 1/2) \vee 0$. Then $Z(g) = \{ x \in X: f(x) \leq \frac{1}{2} \}$.

Let $B = \{ x \in X: f(x) < \frac{1}{2} \}$. Then B is an open set, such that $y \in B \subseteq Z(g)$.

Thus $g \in O_y$. Moreover if $x \in Z(g)$, then $f(x) \leq \frac{1}{2}$ .so $x \notin X \setminus U$.

Hence $Z(g) \subseteq U \subseteq M$. #

Theorem 1.3.22 The maximal ideals in C(X) are precisely the sets:

$M^p = \{ f \in C(X): p \in Cl_{\beta X} Z(f) \}, p \in \beta X$. M^p is fixed if and only if $p \in X$.

For $p \in X$, $M^p = M_p = \{ f \in C(X): f(p) = 0 \}$.

Theorem 1.3.23 The maximal ideals in $C^*(X)$ are precisely the sets:

$M^{*P} = \{f \in C^*(X): f^{\beta}(p) = 0\}$, for all $p \in \beta X$.

Theorem 1.3.24 For each $p \in \beta X$, O^p is the intersection of all prime ideals contained in M^p.

Theorem 1.3.25 Every z-ideal is an intersection of prime ideals.

Theorem 1.3.26 Every prime ideal in C(*X*) is contained in a unique maximal ideal.

(i.e. C(*X*) is a Gelfand ring).

Proof: Let *P* be a prime ideal in C(*X*). Then there is at least one maximal ideal *M* containing *P*. If there exists another maximal ideal *N* containing *P*, then $M \cap N$ is a z-ideal containing *P*, since *M* and *N* are z-ideals. But $M \cap N$ is not prime by 1.1.4. Hence $M \cap N$ cannot contain a prime ideal by 1.3.9, which is a contradiction. Therefore *P* is contained in a unique maximal ideal. #

Chapter 2

The Subring $C_\infty(X)$

In this chapter, we will define the subring $C_\infty(X)$ and investigate its general properties. We also investigate the relation between this subring and the ideal $C_K(X)$.

The material of this chapter can be found in (Gillman and Jerison, 1976) and (Abu Osba, 1999).

2.1 The Subring $C_\infty(X)$

Definition 2.1.1 Let $C_\infty(X)$ denote the family of all functions f in *C (X)* for which the set $\{x \in X : |f(x)| \geq \frac{1}{n}\}$ is compact for every $n \in \mathbb{N}$.

These functions are usually called, functions vanishing at infinity.

Lemma 2.1.2 Let $f, g \in$ C (X). Then

$$\{x \in X: |(f-g)(x)| \geq \frac{1}{n}\} \subseteq \{x \in X: |f(x)| \geq \frac{1}{2n}\} \cup \{x \in X: |g(x)| \geq \frac{1}{2n}\}$$

and $\{x \in X: |(fg)(x)| \geq \frac{1}{n}\} \subseteq \{x \in X: |f(x)| \geq \frac{1}{\sqrt{n}}\} \cup \{x \in X: |g(x)| \geq \frac{1}{\sqrt{n}}\}$,

moreover if $f \in C_\infty(X)$, then $\{x \in X: |f(x)| \geq \frac{1}{\sqrt{n}}\}$ is compact.

Proof: Let $|f(x)| < \frac{1}{2n}$ and $|g(x)| < \frac{1}{2n}$. Then

$$|f(x) - g(x)| \leq |f(x)| + |g(x)| < \frac{1}{2n} + \frac{1}{2n} = \frac{1}{n},$$

so $\{x \in X: |(f-g)(x)| \geq \frac{1}{n}\} \subseteq \{x \in X: |f(x)| \geq \frac{1}{2n}\} \cup \{x \in X: |g(x)| \geq \frac{1}{2n}\}$.

Let $|f(x)| < \frac{1}{\sqrt{n}}$ and $|g(x)| < \frac{1}{\sqrt{n}}$. Then

$$|f(x).g(x)| = |f(x)||g(x)| < \frac{1}{\sqrt{n}} \cdot \frac{1}{\sqrt{n}} = \frac{1}{n}.$$

Thus $\{x \in X: |(fg)(x)| \geq \frac{1}{n}\} \subseteq \{x \in X: |f(x)| \geq \frac{1}{\sqrt{n}}\} \cup \{x \in X: |g(x)| \geq \frac{1}{\sqrt{n}}\}$.

Let $f \in C_\infty(X)$. Then $\{x \in X: |f(x)| \geq \frac{1}{n}\}$ is compact, but

$$\left[\sqrt{n}\right] \leq \sqrt{n} \leq \left[\sqrt{n}\right] + 1 \text{ so } \frac{1}{\left[\sqrt{n}\right]+1} \leq \frac{1}{\sqrt{n}} \leq \frac{1}{\left[\sqrt{n}\right]}.$$

Therefore $\{x \in X: |f(x)| \geq \frac{1}{\sqrt{n}}\} \subseteq \{x \in X: |f(x)| \geq \frac{1}{\left[\sqrt{n}\right]+1}\}$ which is

compact. Hence $\{x \in X: |f(x)| \geq \frac{1}{\sqrt{n}}\}$ is compact. #

Theorem 2.1.3 The set $C_\infty(X)$ is a subring in $C(X)$.

Proof: Let $f, g \in C_\infty(X)$. Then $\{x \in X: |f(x)| \geq \frac{1}{n}\}$ and $\{x \in X: |g(x)| \geq \frac{1}{n}\}$ are

compact. But by 2.1.2,

$$\{x \in X: |(f-g)(x)| \geq \frac{1}{n}\} \subseteq \{x \in X: |f(x)| \geq \frac{1}{2n}\} \cup \{x \in X: |g(x)| \geq \frac{1}{2n}\}$$

and $\{x \in X: |(fg)(x)| \geq \frac{1}{n}\} \subseteq \{x \in X: |f(x)| \geq \frac{1}{\sqrt{n}}\} \cup \{x \in X: |g(x)| \geq \frac{1}{\sqrt{n}}\}$.

Now, since the union of two compact sets is compact it follows that

$\{x \in X: |(f-g)(x)| \geq \frac{1}{n}\}$ and $\{x \in X: |(fg)(x)| \geq \frac{1}{n}\}$ are compact, hence

fg and $f-g \in C_\infty(X)$. Thus $C_\infty(X)$ is a subring in $C(X)$. #

Example 2.1.4

1 – Since $\mathbb{N}$ is a discrete space we have

$$C_\infty(\mathbb{N}) = \{f \in C(\mathbb{N}): |f(x)| \geq \frac{1}{n} \text{ for finitely many elements } x \in \mathbb{N}\}.$$

2 – $C_\infty(\mathbb{Q}) = \{0\}$.

Let $f \in C_\infty(\mathbb{Q})$. Then $\{x \in \mathbb{Q} : |f(x)| \geq \frac{1}{n}\}$ is compact in $\mathbb{Q}$, so $\{x \in \mathbb{Q} : |f(x)| \geq \frac{1}{n}\}$ is compact in $\mathbb{R}$. But $\mathbb{R}$ is T_2 - space, hence $\{x \in \mathbb{Q} : |f(x)| \geq \frac{1}{n}\}$ is closed in $\mathbb{R}$.

Let U be any open set in $\mathbb{R}$ such that

$$U \cap \mathbb{Q} = \{x \in \mathbb{Q} : |f(x)| > \frac{1}{n}\} \subseteq \{x \in \mathbb{Q} : |f(x)| \geq \frac{1}{n}\}, \text{ but}$$

$$\mathrm{Cl}_{\mathbb{R}}(U \cap \mathbb{Q}) = \mathrm{Cl}_{\mathbb{R}}(U \cap \mathrm{Cl}_{\mathbb{R}}\mathbb{Q}) = \mathrm{Cl}_{\mathbb{R}}(U \cap \mathbb{R}) = \mathrm{Cl}_{\mathbb{R}}U.$$

Thus $\mathrm{Cl}_{\mathbb{R}}U = \mathrm{Cl}_{\mathbb{R}}(U \cap \mathbb{Q}) = \mathrm{Cl}_{\mathbb{R}}\{x \in \mathbb{Q} : |f(x)| > \frac{1}{n}\} = \{x \in \mathbb{Q} : |f(x)| \geq \frac{1}{n}\} \subseteq \mathbb{Q}$.

Hence there exists an open interval (a, b) such that

$$(a, b) \subseteq U \subseteq \mathrm{Cl}_{\mathbb{R}}U = \{x \in \mathbb{Q} : |f(x)| \geq \frac{1}{n}\} \subseteq \mathbb{Q}.$$

This is impossible unless $U = \varnothing$ therefore $f = 0$. So $C_\infty(\mathbb{Q}) = \{0\}$. #

3 – $C_\infty(\mathbb{R}) = \{f \in C(\mathbb{R}) : \lim\limits_{|x| \to \infty} f(x) = 0\}$.

Let $f \in C_\infty(\mathbb{R})$ and $\varepsilon > 0$. Then $\{x \in \mathbb{R} : |f(x)| \geq \frac{1}{n}\}$ is compact in $\mathbb{R}$. So there exists

$N \in \mathbb{N}$ such that $\{x \in \mathbb{R} : |f(x)| \geq \frac{1}{n}\} \subseteq [-N, N]$. $\varepsilon > \frac{1}{n}$,

Let $n \in \mathbb{N}$ such that $\varepsilon > \frac{1}{n}$.

Hence for all $x \in \mathbb{R}$ such that $|x| > N$, we have $|f(x)| < \frac{1}{n} < \varepsilon$, thus $\lim\limits_{|x| \to \infty} f(x) = 0$.

Conversely, assume $\lim\limits_{|x| \to \infty} f(x) = 0$. Then $\forall \varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$|f(x)| < \varepsilon$ for all $|x| > N$. So take $\varepsilon = \frac{1}{n}$, hence $|f(x)| < \frac{1}{n}$ for all $|x| > N$.

Therefore $|f(x)| \geq \frac{1}{n}$, for all x $\in [-N,N]$ (i.e $\{x \in \mathbb{R} : |f(x)| \geq \frac{1}{n}\} \subseteq [-N,N]$).

So $\{x \in \mathbb{R} : |f(x)| \geq \frac{1}{n}\}$ is compact. Thus $f \in C_\infty(\mathbb{R})$. #

Perhaps, this is why the functions in $C_\infty(X)$ are called the functions vanishing at infinity.

Lemma 2.1.5 Every function in $C_\infty(X)$ is bounded.

Proof: Let $f \in C_\infty(X)$. Then $\{x \in X : |f(x)| \geq \frac{1}{n}\} = A$ is compact.

Hence $f(A)$ is compact in $\mathbb{R}$, and so there exists $m \in \mathbb{N}$ such that $f(A) \subseteq [-m, m]$.

Now, let $B = \{x \in X : |f(x)| < \frac{1}{n}\}$. Then $f(B) \subseteq [-\frac{1}{n}, \frac{1}{n}]$.

Therefore $f(X) = f(A) \cup f(B) \subseteq [-m, m]$, thus f is bounded. #

Theorem 2.1.6 $C_\infty(X)$ is an ideal in $C^*(X)$.

Proof: Let $f, g \in C_\infty(X)$, and $h \in C^*(X)$. Then

$\{x \in X : |(f-g)(x)| \geq \frac{1}{n}\} \subseteq \{x \in X : |f(x)| \geq \frac{1}{2n}\} \cup \{x \in X : |g(x)| \geq \frac{1}{2n}\}$, but the finite union of compact sets is compact, so $\{x \in X : |(f-g)(x)| \geq \frac{1}{n}\}$ is compact.

Thus $f - g \in C_\infty(X)$.

Assume $a \in \{x \in X : |hf(x)| \geq \frac{1}{n}\}$. Since $h \in C^*(X)$ there exists a positive integer M such that $|h(x)| \leq M$, so $\frac{1}{n} \leq |hf(a)| = |h(a)||f(a)| \leq M\,|f(a)|$, hence $|f(a)| \geq \frac{1}{nM}$, and so $a \in \{x \in X : |f(x)| \geq \frac{1}{nM}\}$.

Therefore $\{x \in X : |hf(x)| \geq \frac{1}{n}\} \subseteq \{x \in X : |f(x)| \geq \frac{1}{nM}\}$.

Hence $\{x \in X : |hf(x)| \geq \frac{1}{n}\}$ is compact because it is a closed subset of a compact set.

Thus $hf \in C_{\infty}(X)$. #

Note that: if $f(x) = \frac{1}{1+x^2} \in C_{\infty}(\mathbb{R})$ and $h(x) = (1+x^2)^2$ (not bounded), then

$hf(x) = 1+x^2 \notin C_{\infty}(\mathbb{R})$.

This shows that $C_{\infty}(\mathbb{R})$ is not an ideal in $C(\mathbb{R})$.

Theorem 2.1.7 The subring $C_{\infty}(X)$ is the intersection of the all free maximal ideals of $C^*(X)$.

Proof: The intersection of all free maximal ideals of $C^*(X)$ coincides with the set

$$I = \bigcap_{P \in \beta X - X} M^{*p} = \{ f \in C^*(X) : f^{\beta}(\beta X - X) = 0 \}.$$

Now, let $f \in I$, then $\beta X - X \subseteq Z(f^{\beta})$, so $\beta X - Z(f^{\beta}) \subseteq X$.

For each $n \in \mathbb{N}$ the set $\{x \in \beta X : |f^{\beta}(x)| \geq \frac{1}{n}\}$ is closed in βX and so it is compact.

But $\{x \in \beta X : |f^{\beta}(x)| \geq \frac{1}{n}\} \subseteq X$,

hence $\{x \in X : |f(x)| \geq \frac{1}{n}\} = \{x \in \beta X : |f^{\beta}(x)| \geq \frac{1}{n}\}$ is compact in X, thus $f \in C_{\infty}(X)$ and $I \subseteq C_{\infty}(X)$.

Conversely, assume that $f \in C_{\infty}(X)$ and $f \notin M^{*p}$, for some $p \in \beta X - X$.

So $f^{\beta}(p) = r > \frac{1}{n}$ for some $n \in \mathbb{N}$.

Let $B_n = \{x \in X : |f(x)| > \frac{1}{n}\}$ and $A_n = Cl_X B_n = \{x \in X : |f(x)| \geq \frac{1}{n}\}$.

But $B_n = \{x \in \beta X : \left|f^{\beta}(x)\right| > \frac{1}{n}\} \cap X$, and $f \in C_\infty(X)$, so A_n is compact in X and in βX . Therefore $A_n = Cl_{\beta X} A_n = Cl_{\beta X} B_n = Cl_{\beta X}(\{x \in \beta X : \left|f^{\beta}(x)\right| > \frac{1}{n}\} \cap X)$

$= Cl_{\beta X}(\{x \in \beta X : \left|f^{\beta}(x)\right| > \frac{1}{n}\} \cap Cl_{\beta X} X) = Cl_{\beta X}\{x \in \beta X : \left|f^{\beta}(x)\right| > \frac{1}{n}\}$

$= \{x \in \beta X : \left|f^{\beta}(x)\right| \geq \frac{1}{n}\}$. Hence $p \in \{x \in \beta X : \left|f^{\beta}(x)\right| \geq \frac{1}{n}\} = A_n \subseteq X$, this is a contradiction.

Thus $f \in M^{*P}$ for all $p \in \beta X - X$, and so $f \in \bigcap_{p \in \beta X - X} M^{*p}$. #

Theorem 2.1.8 $C_\infty(X)$ is a z-ideal in $C^*(X)$.

Proof: It follows from 1.3.8, 1.3.10, and 2.1.7. #

Theorem 2.1.9 The space *X* is compact if and only if $C_\infty(X) = C^*(X)$.

Proof: Let $f \in C^*(X)$ and *X* is compact. But $\{x \in X : \left|f(x)\right| \geq \frac{1}{n}\}$ is a closed subset of *X*, and so $\{x \in X : \left|f(x)\right| \geq \frac{1}{n}\}$ is compact. Hence $f \in C_\infty(X)$. Thus $C_\infty(X) = C^*(X)$.

Conversely, Let *X* be a non-compact space. Then take *f (x)* to be the unity function of the ring $C^*(X)$. So, $X = \{x \in X : \left|f(x)\right| \geq \frac{1}{n}\}$ is non compact. Hence $C_\infty(X)$ is proper in $C^*(X)$. #

Theorem 2.1.10 $C_\infty(X)$ is a free proper ideal in $C^*(X)$ if and only if X is a locally compact non-compact space.

Proof: Suppose $C_\infty(X)$ is a free proper ideal.

Hence *X* is not compact, otherwise $C_\infty(X) = C^*(X)$.

Let $x_0 \in X$. Then there exists $f \in C_\infty(X)$ such that $f(x_0) \neq 0$, since $C_\infty(X)$ is free.

So there exists $n \in \mathbb{N}$ such that $x_0 \in \{x \in X : |f(x)| > \frac{1}{n}\} \subseteq \{x \in X : |f(x)| \geq \frac{1}{n}\}$, which

is a compact neighborhood. Thus *X* is locally compact.

Conversely, assume *X* is a locally compact, non-compact space.

Then $C_\infty(X)$ is a proper ideal of $C^*(X)$, since $1 \notin C_\infty(X)$.

Let $x_o \in X$, U an open neighborhood of $x_\circ$ such that $\overline{U}$ is compact.

Complete regularity of X implies that there exists $f \in C^*(X)$ such that

$f(x_\circ) = 1$, and $f(X - U) = 0$. So, $\{x \in X : |f(x)| > \frac{1}{n}\} \subseteq coz\ f \subseteq U$.

Hence $\{x \in X : |f(x)| \geq \frac{1}{n}\} \subseteq coz\ f \subseteq \overline{U}$ and therefore $\{x \in X : |f(x)| \geq \frac{1}{n}\}$ is

compact. Thus $f \in C_\infty(X)$, and $f(x_\circ) \neq 0$. So $C_\infty(X)$ is a proper free ideal. #

Lemma 2.1.11 Let $f, g \in C_\infty(X)$. Then if there exists

$n_0 \in \mathbb{N}$ such that $\{x \in X : |g(x)| < \frac{1}{n_0}\} \subseteq Z(f)$, then f is a multiple of g in $C_\infty(X)$.

Proof:

Let $h(x) = \begin{cases} \dfrac{f(x)}{g(x)} & |g(x)| \geq \dfrac{1}{2n_0} \\ 0 & |g(x)| \leq \dfrac{1}{2n_0} \end{cases}$.

Then $h \in$ C(X) and $f = gh$, but for every $n \in \mathbb{N}$, we have

$\{x \in X : |h(x)| \geq \frac{1}{n}\} \subseteq \{x \in X : |f(x)| \geq \frac{1}{2n_0 n}\}$ which is compact, and so

$\{x \in X : |h(x)| \geq \frac{1}{n}\}$ is compact for any $n \in \mathbb{N}$ (i.e. $h \in C_\infty(X)$). #

Proposition 2.1.12 For a Hausdroff space *X*, the following statements are equivalent:

(1) *X* is locally compact.

(2) $B = \{X \setminus Z(f) : f \in C_\infty(X)\}$ is a base for open sets in *X*.

(3) The collection $C_\infty(X)$ separates points from closed sets (i.e. if F is a closed set in *X* and $x_0 \notin F$, then there exists $f \in C_\infty(X)$ such that $f(x_0) = 1$ and $f(F)=0$).

Proof: (1)⟶(2) Let G be an open set in X and $x_0 \in G$. Then there exists a compact set H such that $x_0 \in \operatorname{int} H \subseteq H \subseteq G$.

Now define $f \in C(X)$ with $f(x_0) = 1$ and $f(X \setminus \operatorname{int} H) = 0$. Since

$\{x \in X : |f(x)| \geq \frac{1}{n}\} \subseteq X \setminus Z(f) \subseteq H$, so $\{x \in X : |f(x)| \geq \frac{1}{n}\}$ is compact for all $n \in \mathbb{N}$. Hence $f \in C_\infty(X)$ and clearly $x_0 \in X \setminus Z(f) \subseteq G$. Thus B is a base for open sets in *X*.

(2)⟶(3) Clear.

(3)⟶(1) Let G be any open set and $x_0 \in G$. Then there exists $f \in C_\infty(X)$ such that $f(X \setminus G) = 0$ and $f(x_0) = 1$.

Therefore $x_0 \in \{x \in X : |f(x)| \geq \frac{1}{n}\} \subseteq G$.

Now, let $H = \{x \in X : |f(x)| \geq \frac{1}{2}\}$. Then *H* is a compact set and $x_0 \in \operatorname{int} H \subseteq H \subseteq G$.

Hence *X* is locally compact. #

Lemma 2.1.13 Let *A* be an open subset of *X*. Then $A = X \setminus Z(f)$ for some $f \in C_\infty(X)$ if and only if *A* is a σ - compact locally compact subset of *X*.

Proof: Let $A = X \setminus Z(f)$ for some $f \in C_\infty(X)$. Then $A = \bigcup_{n=1}^{\infty} A_n$ where

$A_n = \{x \in X : |f(x)| \geq \frac{1}{n}\}$. Since each A_n is compact, A is σ - compact.

If $x \in A$, there exists $n_0 \in \mathbb{N}$ such that $x \in \{y \in X: |f(y)| > \frac{1}{n_0}\} \subseteq A_{n_0}$. Thus we get A is a locally compact subset of X.

Conversely, let A be a σ - compact locally compact subset of X. Then $A = \bigcup_{n=1}^{\infty} A_n$ with A_n is compact and $A_n \subseteq \operatorname{int} A_{n+1}$. Now for each $n \in \mathbb{N}$, there exists $f_n \in C(X)$ such that $f_n(X) \subseteq [0, 1]$, $f_n(A_n) = \{1\}$, and $f_n(X / \operatorname{int} A_{n+1}) = \{0\}$.

So $f = \sum_{n=1}^{\infty} \frac{f_n}{2^n}$ is an element of C(X), by the Weierstrass M-test. Hence $A = X / Z(f)$.

Let $x_0 \notin A_{n+1}$. Then $f_1(x_0) = \ldots = f_n(x_0) = 0$ and so

$f(x_0) \leq (1/2^{n+1}) + \ldots \leq (1/2^n) < (1/n)$. So $x_0 \notin \{x \in X: |f(x)| \geq \frac{1}{n}\}$, and hence $\{x \in X: |f(x)| \geq \frac{1}{n}\} \subseteq A_{n+1}$ and so we get $f \in C_\infty(X)$. #

Lemma 2.1.13 gives a new representation for $Z[C_\infty(X)]$. If F is the collection of all closed subsets of X, then

$Z[C_\infty(X)] = \{H \in F: X \setminus H$ is a locally compact σ - compact set$\}$.

Lemma 2.1.14 No point of $A \subseteq X$ has a compact neighborhood in X if and only if $f(A) = \{0\}$ for all $f \in C_\infty(X)$.

Proof: If $a \in A$ and $f(a) \neq 0$, for some $f \in C_\infty(X)$, then there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < |f(a)|$ and hence $H = \{x \in X: |f(x)| \geq \frac{1}{n+1}\}$ is a compact neighborhood of a, a contradiction.

Conversely, suppose the point a has a compact neighborhood H. Then there exists $f \in C(X)$ such that $f(a) = 1$ and $f(X / \operatorname{int} H) = \{0\}$.

Hence for every $n \in \mathbb{N}$, we have $\{x \in X: |f(x)| \geq \frac{1}{n}\}$ □ H.

Thus the closed set $\{x \in X : |f(x)| \geq \frac{1}{n}\}$ is compact and so $f \in C_\infty(X)$. #

Definition 2.1.15 Let X_L denotes the set of all points of X with compact nbhds. It is clear that, X is locally compact if and only if $X = X_L$.

Theorem 2.1.16 For each space X, $X_L = coz\ C_\infty(X) = \bigcup_{f \in C_\infty(X)} coz\ f$.

Proof: Let $x \in coz\ C_\infty(X)$. Then there exists $f \in C_\infty(X)$ such that $f(x) \neq 0$, which implies that there exists $n \in \mathbb{N}$ such that $|f(x)| > \frac{1}{n}$. So

$x \in \{ y \in X: |f(y)| > \frac{1}{n}\} \subseteq \{ y \in X: |f(y)| \geq \frac{1}{n}\}$ which is a compact nbhd of x.

Hence $x \in X_L$.

Conversely suppose $x \in X_L$, and suppose U is an open nbhd of x such that $\overline{U}$ is compact. Then there exists $f \in C(X)$ such that $f(x) = 1$ and $f(X - U) = 0$.

So $\{x \in X: |f(x)| > \frac{1}{n}\} \subseteq coz\ f \subseteq U$ for each $n \in \mathbb{N}$, and therefore

$\{x \in X: |f(x)| \geq \frac{1}{n}\} \subseteq \overline{U}$. Hence $\{x \in X: |f(x)| \geq \frac{1}{n}\}$ is compact.

Thus $x \in coz\ f$, and $f \in C_\infty(X)$. i.e. $x \in coz\ C_\infty(X)$. So $X_L = coz\ C_\infty(X)$. #

Theorem 2.1.17 $C_\infty(X)$ is faithfull (i.e. $Ann\ (C_\infty(X)) = \{0\}$) if and only if X_L is dense in X.

Proof: Suppose X_L is not dense in X.

Let $x \in X - \overline{X_L}$.

Let $f \in C(X)$ such that $f(x) = 1$ and $f(\overline{X_L}) = 0$.

Then $f \neq 0$, and $f \in Ann\ (C_\infty(X))$, since for each $g \in C_\infty(X)$, $f(coz\ g) = 0$,

so $f\,g = 0$.

Conversely, suppose $\overline{X_L} = X$. Let $f \in Ann\,(C_\infty(X))$. Then $f\,(coz\,g\,) = 0$,

for each $g \in C_\infty(X)$, since $f\,g = 0$. So $f\,(X_L) = 0$.

Hence $f = 0$ since $\overline{X_L} = X$. Thus $Ann\,(C_\infty(X)) = \{0\}$. #

Examples 2.1.18 Here we calculate X_L for some well known spaces X.

1- $\mathbb{R}_L = \mathbb{R}$ and $\mathbb{N}_L = \mathbb{N}$ since they are locally compact spaces, while $\mathbb{Q}_L = \varnothing$.

2- Let $X = [-1, 1]$ with all points are isolated except for the point $\{0\}$ having its usual nbhd. Then $X_L = X - \{0\}$ is dense in X.

Definition 2.1.19 A non-zero ideal E in a ring R is called essential ideal if it intersects every non-zero ideal non-trivially.

Definition 2.1.20 The subset D of X is called nowhere dense in X if $Int_X Cl_X D = \varnothing$.

Theorem 2.1.21 Let E be a non-zero ideal in $C(X)$. Then the following statements are equivalent:

1- E intersects every non-zero z-ideal in $C(X)$ non-trivially.

2- E is essential ideal in $C(X)$.

3- $Ann\,(E) = (0)$.

4- $\bigcap Z[E]$ is a nowhere dense subset of X.

Proof: See (Al-Zaghal, 2006). #

Theorem 2.1.22 Let $C_\infty(X)$ be an ideal in $C(X)$. Then $C_\infty(X)$ is an essential ideal if and only if X_L is dense in X.

Proof: The result follows from 2.1.17 and 2.1.21. #

Lemma 2.1.23 Every free ideal is an essential ideal.

Proof: Let E is a free ideal. Then $\bigcap Z[E] = \varnothing$.

Hence $Int \bigcap Z[E] = \varnothing,$ and so E is an essential ideal. #

Definition 2.1.24 A space *X* is called an almost compact space if β X is the one point compactification of *X* (i.e. $\beta X = X^* = X \cup \{\infty\}$).

It was shown in 2.1.4 that $C_\infty(\mathbb{Q}) = \{0\}$ is a non-maximal ideal in $C(\mathbb{Q})$.

Theorem 2.1.25 The following statements are equivalent:

(1) $C_\infty(X)$ is a maximal ideal in $C^*(X)$.

(2) *X* is an almost compact space.

Proof: (1) $\to$(2) Suppose $C_\infty(X)$ is a maximal ideal in $C^*(X)$.

Then *X* is an almost compact space, since $C_\infty(X) = \bigcap_{P \in \beta X - X} M^{*P} = M^{*P}$.

Hence $|\beta X - X| = 1,$ so *X* is almost compact space.

(2) $\to$ (1) Suppose *X* is almost compact. Then $\beta X = X \bigcup \{\infty\}$, and so $C_\infty(X) = M^{*p}$. Hence $C_\infty(X)$ is maximal ideal in $C^*(X)$.

Example 2.1.26 Let $X = \beta\mathbb{R} - \{0\}$. Then $\beta X = X \bigcup \{0\} = X^*$. So *X* is an almost compact space and $C_\infty(X)$ is a maximal ideal in $C^*(X)$.

2.2 The ideal $C_K(X)$

In this section, we study the ideal $C_K(X)$ which is very related to $C_\infty(X)$.

Definition 2.2.1 Let $C_K(X)$ denotes the family of all functions in *C (X)* having compact support.

Theorem 2.2.2 $C_K(X)$ is an ideal in *C*(*X*).

Proof: Let $f, g \in C_K(X)$. Then $Z(f) \cap Z(g) \subseteq Z(f-g)$.

So $coz(f-g) \subseteq coz\ f \cup coz\ g$.

Thus $\text{supp}(f-g) \subseteq \overline{coz\ f \cup coz\ g} = \text{supp}\, f \cup \text{supp}\, g$, which is compact, because a finite union of compact sets is compact. So $f - g \in C_K(X)$.

Now, let $h \in C(X)$, $f \in C_K(X)$. Then $Z(hf) = Z(f) \cup Z(h)$, hence

$coz(hf) = coz\ f \cap coz\ h \subseteq coz\ f$, and so, $\text{supp}(hf) \subseteq \text{supp}\, f$, and therefore it is compact.

Thus $hf \in C_K(X)$, and so $C_K(X)$ is an ideal of *C*(*X*).

Let $f \in C_K(X)$. Then $f_1 = f_{|\text{supp} f}$ is a continuous real valued function defined on a compact set. So f_1 is bounded. Hence f is bounded, and so $C_K(X) \subseteq C^*(X)$. #

Theorem 2.2.3 $C_K(X)$ is a free proper ideal if and only if *X* is locally compact but not compact.

Proof: Suppose that $C_K(X)$ is a free proper ideal.

Hence X is not compact, otherwise $C_K(X) = C(X)$ is not proper.

Let $x \in X$. Then there exists $f \in C_K(X)$, such that $f(x) \neq 0$, since $C_K(X)$ is free.

So $x \in coz\ f \subseteq \text{supp}\, f$, which is a compact neighborhood. Thus X is locally compact.

Conversely, suppose that X is a locally compact, non-compact space.

Then $C_K(X)$ is a proper ideal of C(X), since $1 \in C(X) - C_K(X)$.

Let $x \in X$, U an open neighborhood of x such that $\overline{U}$ is compact.

Complete regularity of X implies that there exists $f \in C(X)$ such that

$f(x) = 1$, and $f(X - U) = 0$. So coz $f \subseteq U$.

Hence supp $f \subseteq \overline{U}$ which is compact. Thus $f \in C_K(X)$, and x $\in$ coz f.

So $C_K(X)$ is a proper free ideal. #

Theorem 2.2.4 For any space *X*, $C_K(X) = O^{\beta X - X} = \bigcap_{p \in \beta X - X} O^p$.

Proof: If $f \in C_K(X)$, then supp f is compact in *X*.

Now, since $X \subseteq \beta X$, then supp f is compact in βX, but βX is T_2 - space, so

supp *f* is closed in βX.

For each p$\in \beta X - X$, p$\in V = \beta X -$ supp *f*, an open subset of βX.

So $V \cap X = (\beta X -$ supp $f) \cap X = X -$ supp $f \subseteq Z(f)$.

Hence $p \in V \cap X \subseteq Z(f)$. Hence by 1.3.18, $f \in O^p$, for each p$\in \beta X - X$.

Thus $C_K(X) \subseteq \bigcap_{P \in \beta X - X} O^p$.

Now, suppose that $f \in O^{\beta X - X}$, and x $\in \beta X$ - supp *f*.

If x $\in X$, then x $\in X -$ supp *f*, and so there exists an open set *V* of βX containing *x*

such that $V \cap X = X -$ supp $f = \text{Int}_X Z(f)$.

Hence $x \in V \cap X \subseteq Z(f)$. Thus *coz* $f \subseteq X - (V \cap X)$.

Therefore supp $f \subseteq X - (V \cap X)$, so $(V \cap X) \cap$ supp $f = \varnothing$, hence $V \cap$ supp $f = \varnothing$.

Now, since supp $f \subseteq X$, thus x$\in V \subseteq \beta X$ - supp *f*,

If x $\in \beta X - X$, then since $f \in O^{\beta X - X}$ there exists an open set *V* of βX containing *x*

such that $V \cap X \subseteq Z(f)$. So *coz* $f \subseteq X - (V \cap X)$.

Thus supp $f \subseteq X - (V \cap X)$, therefore $(V \cap X) \cap$ supp $f = \varnothing$,

hence $V \cap \operatorname{supp} f = \emptyset$, since $V = (V \cap X) \cup (V \cap \beta X - X)$ and $\operatorname{supp} f \subseteq X$.

So $\operatorname{supp} f \cap (V \cap \beta X - X) = \emptyset$, and $\operatorname{supp} f \cap (V \cap X) = \emptyset$.

So $V \cap \operatorname{supp} f = \emptyset$. Thus $x \in V \subseteq \beta X - \operatorname{supp} f$. Hence $\beta X - \operatorname{supp} f$ is an open set

in βX and so $\operatorname{supp} f$ is close in βX. Thus $\operatorname{supp} f$ is compact and $f \in C_K(X)$.

So $\bigcap_{P \in \beta X - X} O^p \subseteq C_K(X)$, therefore $C_K(X) = O^{\beta X - X} = \bigcap_{P \in \beta X - X} O^p$. #

Lemma 2.2.5 For each $p \in X$, $\bigcap Z[O_p] = \{p\}$.

Proof: Let $p \neq q$. Then there exists disjoint open sets U, V in X such that $p \in U$ and $q \in V$. Since X is completely regular so there exists an open set K such that

$p \in K \subseteq \overline{K} \subseteq U \subseteq \overline{U}$ and $V \cap \overline{U} = \emptyset$.

Define $f: X \longrightarrow \mathbb{R}$ such that $f(\overline{K}) = 0$ and $f(q) = 1$. Then $f \in O_P$,

hence $q \notin Z(f) \in Z[O_p]$ and clearly $p \in \bigcap Z[O_p]$ therefore $\bigcap Z[O_p] = \{p\}$. #

Theorem 2.2.6 $C_K(X)$ is the intersection of all free ideals in $C(X)$.

Proof: For each $p \in \beta X - X$, O^p is a free ideal, since $\bigcap Z[O^p] = \emptyset$.

So the intersection of all free ideals is contained in $O^{\beta X - X} = C_K(X)$.

Now, let I be any free ideal of $C(X)$. Hence for every compact set A, there exists $g \in I$ such that g has no zeros in A, see (Gillman and Jerison, 1976).

So there exists $g \in I$ such that g has no zero in supp f,

i.e. $\operatorname{supp} f \subseteq coz\, g$.

Define $h(x) = \begin{cases} \dfrac{f}{g}(x) & x \in \operatorname{supp} f \\ 0 & \text{otherwise} \end{cases}$

Then $h \in C(X)$, and $f = hg \in I$.

Hence $C_K(X) \subseteq I$, for every free ideal I of $C(X)$.

So $C_K(X) = \bigcap_{I \text{ free ideal in C(X)}} I$.   #

Theorem 2.2.7 $C_K(X) \subseteq C_\infty(X)$, and the inclusion may be proper.

Proof: Let $f \in C_K(X)$.

Now, for each $n \in \mathbb{N}$, the set $\{x \in X: |f(x)| > \frac{1}{n}\} \subseteq coz\, f$.

So the closed set $\{x \in X: |f(x)| \geq \frac{1}{n}\}$ is contained in supp f.

Hence for each $n \in \mathbb{N}$, the set $\{x \in X: |f(x)| \geq \frac{1}{n}\}$ is compact, and so $f \in C_\infty(X)$.

Thus $C_K(X) \subseteq C_\infty(X)$.

Now, let $j: \mathbb{N} \longrightarrow \mathbb{R}$ be defined such that $j(n) = 1/n$ for each $n \in \mathbb{N}$. Then

$j \in C(\mathbb{N})$, and supp $j = \mathbb{N}$ is non-compact so $j \notin C_K(\mathbb{N})$.

For each $m \in \mathbb{N}$ the set $\{n \in \mathbb{N}: |j(n)| \geq 1/m\} = \{1, 2, 3, \ldots, m\}$ is compact.

Therefore $j \in C_\infty(\mathbb{N}) - C_K(\mathbb{N})$. Thus $C_\infty(\mathbb{N}) \neq C_K(\mathbb{N})$. #

The above example shows that $C_\infty(X)$ needs not to be an ideal in C(X), since the function j is a unit and belongs to the proper set $C_\infty(\mathbb{N})$.

Theorem 2.2.8 The following statements are equivalent:

(1) $C_\infty(X) = \{0\}$.

(2) $C_K(X) = \{0\}$.

(3) No point in X has a compact neighborhood.

(4) $\beta X - X$ is dense in βX .

Proof: (1) $\rightarrow$(2) Since $C_K(X) \subseteq C_\infty(X)$, it follows that if $C_\infty(X) = \{0\}$, then $C_K(X) = \{0\}$. So the implication (1)$\rightarrow$(2) is clear.

(2) $\rightarrow$(3) suppose $C_K(X) = \{0\}$. Suppose $x \in X$, has a compact nbhd in X.

So there exists an open set U containing x such that $\overline{U}$ is compact. Now, let

$f \in C(X)$ such that $f(x) = 1$, and $f(X-U) = 0$. So supp $f \subseteq \overline{U}$. Hence $f \in C_K(X)$

and $f \neq 0$, a contradiction.

(3) $\rightarrow$(4) suppose $\beta X - X$ is not dense set in βX.

So there exists an open set U in βX such that $U \cap (\beta X - X) = \emptyset$. i.e. $U \subseteq X$.

Let $x \in U$, then there exists an open set V in βX, such that

$x \in V \subseteq \mathrm{Cl}_{\beta X} V \subseteq U \subseteq X$. But $\mathrm{Cl}_{\beta X} V$ is a compact nbhd of x, since it is closed set

in βX, so x has a compact nbhd, which is a contradiction.

(4) $\rightarrow$(1) Suppose $\beta X - X$ is dense in βX.

Let $f \in C_\infty(X) = \bigcap_{p \in \beta X - X} M^{*p}$, by 2.1.7, so $f^\beta(\beta X - X) = 0$.

Therefore $f^\beta = 0$, since $\beta X - X$ is dense in βX. So $f^\beta|_X = 0$.

Hence $C_\infty(X) = \{0\}$. #

Example 2.2.9

1 – $C_K(\mathbb{N}) = \{f \in C(\mathbb{N}) : f = 0$ except on a finite set$\} \subset C_\infty(\mathbb{N})$.

2 – $C_K(\mathbb{Q}) = 0$, since $C_\infty(\mathbb{Q}) = 0$, see 2.1.4.

3 – $C_K(\mathbb{R}) = \{f \in C(\mathbb{R}) : coz\, f$ is bounded$\} \subset C_\infty(\mathbb{R})$, since the function

$f(x) = \dfrac{1}{1+x^2} \in C_\infty(\mathbb{R}) - C_K(\mathbb{R})$, see 2.1.4.

4 – Let S be the Sorgenfrey line. Then $C_\infty(S) = C_K(S) = \{0\}$, since compact sets in the Sorgenfrey line are countable sets, see (Elpelie, 1976). But a compact nbhd must contain a set of the form [a, b) which is uncountable. So no compact set in S can be a nbhd unless it is empty.

Theorem 2.2.10 For locally compact non – compact space *X:*

1- $C_K(X) \cong O_\infty$, O_∞ is an ideal in $C(X^*)$.

2- $C_\infty(X) \cong M_\infty$ in $C(X^*)$, where $M_\infty = \{f \in C(X) : f(\infty) = 0\}$.

Proof: Let $X^* = X \cup \{\infty\}$ be the one point compactification of *X*.

Observe that is O_∞ a subring of $C(X^*)$.

1 - Let $\psi : O_\infty \longrightarrow C_K(X)$ such that $\psi(f) = f_{|_X}$. Then ψ is well – defined.

Because *X* is dense in X^*, and if $f \in O_\infty$, then there exists a compact set *A* in *X* such that $(X - A) \cup \{\infty\} \subseteq Z(f)$.

So $X - A \subseteq Z(f_{|_X})$, which implies that $coz\ f_{|_X} \subseteq A$.

Hence $f_{|_X} \in C_K(X)$. Let $f, g \in O_\infty$. Then

$\psi(f+g) = (f+g)_{|_X} = f_{|_X} + f_{|_X} = \psi(f) + \psi(g)$, and $\psi(fg) = f_{|_X} \cdot g_{|_X} = \psi(f)\ \psi(g)$. So ψ is ring homomorphism.

If $\psi(f) = 0$, then $f_{|_X} = 0$. Thus $f = 0$, since *X* is dense in X^*.

Now, suppose $f \in C_K(X)$.

Define $f_1(x) = \begin{cases} f(x) & x \in \operatorname{supp} f \\ 0 & x \in X^* - coz\ f \end{cases}$

Then $f_1 \in C(X^*)$, since supp *f* is compact in *X*, and so it is closed in X^*, and *coz f* is open in X^*, since *X* is open in X^*.

But $\{\infty\} \cup (X - \operatorname{supp} f) \subseteq \{\infty\} \cup (X - coz\ f) = X^* - coz\ f \subseteq Z(f_1)$.

So $f_1 \in O_\infty$, and $f_{|_X} = f = \psi(f_1)$. Hence ψ is a ring isomorphism. #

Observe that is M_∞ a subring of $C(X^*)$.

2 - Let $\psi : M_\infty \longrightarrow C_\infty(X)$ such that $\psi(f) = f_{|_X}$. Then

$f_{|_X} \in C_\infty(X)$, since if $f \in M_\infty$, $\{\infty\} \subseteq Z(f) \subseteq \{x \in X^* : |f(x)| < \frac{1}{n}\}$, and so

$\{x \in X : |f_{|_X}(x)| > \frac{1}{n}\} \subseteq X$.

Now, since X is dense in X^*, $\{x \in X : |f_{|_X}(x)| \geq \frac{1}{n}\} \subseteq X^*$. Thus $\{x \in X : |f_{|_X}(x)| \geq \frac{1}{n}\}$ is compact.

Let $f, g \in M_\infty$. Then $\psi(f+g) = (f+g)_{|_X} = f_{|_X} + g_{|_X} = \psi(f) + \psi(g)$, and

$\psi(fg) = f_{|_X} \cdot g_{|_X} = \psi(f)\,\psi(g)$. So ψ is ring homomorphism.

If $\psi(f) = 0$, then $f_{|_X} = 0$. Thus $f = 0$, since X is dense in X^*.

Let $f \in C_\infty(X)$ and $f_1 = f$. Then by 2.1.5, there exists m $\in \mathbb{N}$ such that $|f_1| \leq$m, let

$r_n = \frac{m}{2}(\frac{2}{3})^n$. Then $|f_1| \leq m = 3r_1$, inductively given $f_n \in C_\infty(X)$ with

$|f_n| \leq 3\, r_n$. Define $B_n = \{x \in X : |f_n(x)| \geq r_n\}$ and A_n be a compact

neighborhood of ∞ in the set $\{\infty\} \cup (X - B_n)$.

So A_n and B_n are compact disjoint in X^*, hence there exists $g_n \in C^*(X^*)$ such

that $g_n(A_n) = 0$ and $g_n(B_n) = r_n$ with $|g_n| \leq r_n$.

Let $f_{n+1} = f_n - g_n|_X$. Then $f_{n+1} \in C_\infty(X)$ since

$f_n \in C_\infty(X)$ and $g_n|_X \in C_K(X) \subseteq C_\infty(X)$, so $|f_{n+1}| = |f_n - g_n|_X| \leq 2r_n$.

Let $g = \sum_{i=1}^{n} g_i$. Then $g \in C(X^*)$, $g(\infty) = 0$ and $(g_1 + g_2 + \ldots + g_n)_{|_X} = f_1 - f_{n+1}$.

Now, $\psi(g) = g_{|_X} = f_1 = f$ since $f_n \longrightarrow 0$, thus $g \in M_\infty$ and $g_{|_X} = f_1 = f$.

So ψ is a ring isomorphism. #

Definition 2.2.11 A space X is called a P-space if every G_δ - set in X is open.

Definition 2.2.12 A point p is called a P-point if $O^p = M^p$.

Theorem 2.2.13 The following statements are equivalent:

(1) A space X is a P-space

(2) $O_p = M_p$ for each $p \in X$.

(3) Every zero set is open.

Proof: See (Gillman and Jerison, 1976). #

Theorem 2.2.14 Let X be a locally compact non – compact space. Then $C_K(X) = C_\infty(X)$ if and only if every σ - compact subset of X is contained in a compact set in X.

Proof: Suppose that $C_K(X) = C_\infty(X)$, then $O_\infty = M_\infty$. Since $C_\infty(X) \cong M_\infty$, $C_K(X) \cong O_\infty$ and $O_\infty \subseteq M_\infty$. So ∞ is a P- point, and therefore it follows that any G_δ - set containing ∞ is neighborhood of it by 2.2.13.

Let $A = \bigcup_{n=1}^{\infty} A_n$, where A_n is compact for each n. $X^* - A = \bigcap_{n=1}^{\infty} (X - A_n) \cup \{\infty\}$.

This is a G_δ - set containing ∞, so there exists a compact set B in X such that $(X - B) \cup \{\infty\} \subseteq X^* - A$. Hence $X - B \subseteq X - A$, and so it follows that $A \subseteq B$.

Conversely, suppose every σ - compact set in X is contained in compact set in X.

Let $f \in C_\infty(X)$. Then $coz\, f = \bigcup_{n \in \mathbb{N}} \{x \in X : |f(x)| \geq \frac{1}{n}\}$.

$coz\, f$ is a σ - compact set. Thus there exists a compact set B in X such that $coz\, f \subseteq B$. Therefore supp $f \subseteq B$, and so it follows that $f \in C_K(X)$. #

Theorem 2.2.15 If X is locally compact non – countably compact space, then $C_K(X) \neq C_\infty(X)$. Similarly if X is a σ - compact, non- compact space.

Proof: Suppose $C_K(X) = C_\infty(X)$.

Let $A = \{x_1, x_2, \ldots\}$ be a countable infinite set in X.

Then A is a σ - compact set, therefore it is contained in a compact set B, by 2.2.14.

Now, let U be an open set in A and $x \in A$. Then

$(U \cap B) \setminus \{x\} \cap A \neq \emptyset$, so $U \setminus \{x\} \cap A \neq \emptyset$. Hence x is a cluster point of A in X.

So X is countably compact, which is a contradiction. #

Example 2.2.16: It was shown in 2.2.9 that $C_\infty(\mathbb{R}) \neq C_K(\mathbb{R})$. Note that the space $\mathbb{R}$ is locally compact and σ - compact but it is not compact nor countably compact and the σ - compact subset $(-\infty, \infty) = \bigcup_{n=1}^{\infty} [-n, n]$ is not contained in a compact set.

Chapter 3

When $C_\infty(X)$ is an ideal in C(X) ?

In this chapter we will study when $C_\infty(X)$ is an ideal in $C(X)$ and investigate the relation between this subring and some ideals in $C(X)$.

The material of this chapter can be found in (Azarpanah and Soundararajan, 2001).

3.1 When $C_\infty(X)$ is an ideal in $C(X)$?

First, we give the following example that in general $C_\infty(X)$ need not be an ideal in C(X).

Example 3.1.1 Let $f:\mathbb{R}\longrightarrow\mathbb{R}$ be the function defined by $f(x)=\dfrac{1}{1+x^2}$. Then $f\in C_\infty(\mathbb{R})$.

Let $g:\mathbb{R}\longrightarrow\mathbb{R}$ be the function defined by $g(x)=1+x^2$. Then $g\in$ C($\mathbb{R}$).

So $fg = 1 \notin C_\infty(\mathbb{R})$. Hence, $C_\infty(\mathbb{R})$ is not an ideal in C($\mathbb{R}$).

Definition 3.1.2 Let *A* be a subset of a topological space *X*. Then *A* is said to be bounded in X if for every $g\in C(X)$, there exists $\lambda_g\in\mathbb{R}^+$ such that $|g(x)|<\lambda_g$ for all $x\in A$.

Note that $\mathbb{N}$ is bounded in $\beta\mathbb{N}$ but it is unbounded in $\mathbb{R}$.

Theorem 3.1.3 The following conditions are equivalent:

1- $C_\infty(X)$ is an ideal in $C(X)$.

2- Every open locally compact subset of *X* is bounded.

3- Every open locally compact σ- compact subset of *X* is bounded.

Proof: (1) $\longrightarrow$ (2) Let *Y* be an open locally compact subset of *X*. Suppose that *Y* is unbounded. Then there exists $g\in C(X)$, $g\geq 0$ and points $a_n\in Y$ exist such that

$g(a_n) \geq 2^n$ for all n $\in \mathbb{N}$. We can also assume that $g(a_{n+1}) > g(a_n) + 1$, for every n $\in \mathbb{N}$. Since *Y* is locally compact and open in *X*, for each n $\in \mathbb{N}$, there exists an open set A_n in *X* such that $a_n \in A_n$ and $\mathrm{Cl}\, A_n \subseteq Y$ and $\mathrm{Cl}\, A_n$ is compact.

Put $U_n = g^{-1}\{(g(a_n) - (1/4), g(a_n) + (1/4))\} \cap A_n$. Since $\mathrm{Cl}\, U_n \subseteq \mathrm{Cl}\, A_n$, therefore $\mathrm{Cl}\, U_n$ is compact. If $m \neq n$, then $\mathrm{Cl}\, U_m \cap \mathrm{Cl}\, U_n = \varnothing$.

Now, for every n $\in \mathbb{N}$, choose an open set V_n in *X* such that $a_n \in V_n \subseteq \mathrm{Cl}\, V_n \subseteq U_n$.

For every n $\in \mathbb{N}$, we define $f_n \in$ C(X), $0 \leq f_n \leq 1$, $f_n(\mathrm{Cl}\, V_n) = \{1\}$ and $f_n(X \setminus U_n) = \{0\}$. Let $f = \sum_{n=1}^{\infty} \frac{f_n}{2^n}$, then by the Weierstrass M-test, $f \in$ C(X).

To show that $f \in C_\infty(X)$, let $n_0 \in \mathbb{N}$ and $K = \mathrm{Cl}\, U_1 \cup \ldots \cup \mathrm{Cl}\, U_{n_0}$. Clearly *K* is compact and if $x \in X \setminus K$, then $f(x) = \sum_{n > n_0} \frac{f_n(x)}{2^n} < (\frac{1}{n_0})$.

Thus $\{x \in X: |f(x)| \geq \frac{1}{n_0}\} \subseteq K$ and so is compact. Hence $f \in C_\infty$(X).

Now, we claim that $fg \notin C_\infty$ (X). Let $C = \{x \in X: (f\ g)(x) \geq 1\}$, since $(fg)(a_n) = f(a_n)\, g(a_n) \geq 1$, for all n $\in \mathbb{N}$, then $\{a_n\} \subseteq C$. But $g \in$ C(X) is unbounded on $\{a_n\}$ implying that *C* cannot be compact, i.e., $fg \notin C_\infty$ (X). Thus $C_\infty(X)$ is not an ideal in C(X) which is a contradiction. Hence Y is bounded. #

(2) $\longrightarrow$ (3) Clear.

(3) $\longrightarrow$ (1) Since $C_\infty(X)$ is a subring of C(*X*), it is enough to prove that $fg \in C_\infty(X)$ for every $f \in$ C(X) and any $g \in C_\infty(X)$. By 2.1.13, $X \setminus Z(g) = Y$ is an open locally compact σ - compact subset of *X* and hence, by (3), $f(Y)$ is a bounded subset of $\mathbb{R}$. Now it is easy to see that $g^{1/3} \in C_\infty(X)$, since $g \in C_\infty(X)$.

Moreover, $Z(g^{1/3}) = Z(g)$ implies that $(f\,g^{1/3})(X) = (f\,g^{1/3})(Y) \cup \{0\}$. Since $f(Y)$ is a bounded set in $\mathbb{R}$ and $g^{1/3} \in C_\infty(X)$ is bounded function on X, we get $(f\,g^{1/3})(Y)$ is a bounded set in $\mathbb{R}$ implying that $f\,g^{1/3}$ is bounded on X and so belongs to $C^*(X)$.

Since $C_\infty(X)$ is a ring, $g^{2/3} \in C_\infty(X)$. However, if $h \in C^*(X)$ and $h \in C_\infty(X)$, then $hk \in C_\infty(X)$, by 2.1.6. Therefore $f\,g = (f\,g^{1/3})\,g^{2/3} \in C_\infty(X)$, thus (1) holds. #

Theorem 3.1.4 Let X be a locally compact Hausdorff space. Then $C_\infty(X)$ is an ideal in C(X) if and only if X is pseudocompact space.

Proof: Suppose X is a pseudocompact space. Then $C(X) = C^*(X)$, and so $C_\infty(X)$ is an ideal in C(X) by 2.1.6. Thus $C_\infty(X)$ is an ideal of C(X).

Conversely, if $C_\infty(X)$ is an ideal of C(X), X itself is an open locally compact set and so by 3.1.3, X is bounded, thus X is a pseudocompact space.

We have shown in 2.1.4 that $C_\infty(\mathbb{Q}) = \{0\}$ is an ideal in $C(\mathbb{Q})$, but $\mathbb{Q}$ is not pseudocompact space. This does not contradict Theorem 3.1.4, since $\mathbb{Q}$ is not locally compact.

Note that if $X = [0, \Omega)$ is the set of all ordinal numbers less than the first uncountable ordinal Ω. Then X is pseudocompact space and $C_\infty(X)$ is an ideal in C(X).

Definition 3.1.5 For each space X, let $C_F(X) = \{f \in C(X)\colon coz\, f \text{ is finite}\}$.

$C_\psi(X) = \{ f \in C(X)\colon \text{supp}\, f \text{ is pseudocompact}\}$.

Note that $C_\psi(X)$, and $C_F(X)$ are ideals in C(X).

Theorem 3.1.6 $C_\infty(X) \subseteq C_\psi(X)$ if and only if every open locally compact subset of X is bounded, i.e., $C_\infty(X)$ is an ideal in C(X).

Proof: Assume $C_\infty(X) \subseteq C_\psi(X)$. By 3.1.3, it is enough to show that every open locally compact σ-compact subset A of X is bounded. By 2.1.13, $A = X \setminus Z(f)$

for some $f \in C_\infty(X)$. Now, by our hypothesis, $f \in C_\psi(X)$, so Cl $(X \setminus Z(f))$ is pseudocompact and hence $A = X \setminus Z(f)$ is bounded.

Conversely, assume every open locally compact subset of X be bounded and $f \in C_\infty(X)$. Then $X \setminus Z(f)$ is an open locally compact σ*-compact* set by 2.1.13, hence $X \setminus Z(f)$ is bounded. Therefore, Cl $(X \setminus Z(f))$ is bounded, hence Cl $(X \setminus Z(f))$ is pseudocompact, Thus $f \in C_\psi(X)$. #

Note that if $C_\infty(X) = C_F(X)$, $C_\infty(X) = C_\psi(X)$, or $C_\infty(X) = C_K(X)$, then $C_\infty(X)$ is an ideal in C(X).

Theorem 3.1.8 Suppose that there exists $h \in C_\infty(X)$ with $Z(h)$ Lindelof and bounded. If $C_\infty(X)$ is an ideal in C(X), then X is a compact space.

Proof: By 2.1.13, $X \setminus Z(h)$ is an open locally compact σ-compact subset of X. Hence by

3.1.3, $X \setminus Z(h)$ is bounded. Now if $f \in C(X)$, then $f|_{X/Z(h)}$ is bounded and also $f|_{Z(h)}$ is bounded. Thus f is a bounded function and hence we get X a pseudocompact space. $X = (X \setminus Z(h)) \cup Z(h)$ yields X is a Lindelof space. Since X is nowboth Lindelof and pseudocompact, hence X is a compact space. #

Definition 3.1.9 Let X be a topological space. Then X is called ∞-compact if $C_\infty(X) = C_K(X)$.

Example 3.1.10 The set of all rational numbers and the Sorgenfry line are ∞-compact.

Proposition 3.1.11 Let X be locally compact, ∞-compact space. Then X is pseudocompact.

Proof: Let X be locally compact, ∞-compact. Then $C_\infty(X) = C_K(X)$, so $C_\infty(X)$ is an ideal in C(X). Hence X is pseudocompact, by 3.1.4. #

Definition 3.1.12 Let X be a topological space. Then X is called a pseudo-finite space if every compact subset in X is finite and X is called a pseudo-discrete space if every compact subset of X has finite interior.

Example 3.1.13 The set of all rational numbers with usual topology is pseudo-discrete which is not pseudo-finite. Note that the set $A = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$ is a compact infinite set in $\mathbb{Q}$ with $\operatorname{int} A = \varnothing$.

This example shows that the pseudo-discrete space need not be pseudo-finite space.

Theorem 3.1.14 Every locally compact pseudo-discrete space is discrete space.

Proof: Let X be a locally compact pseudo-discrete space and $x \in X$. Then x has a nbhd U whose closure is compact. Hence Int (Cl (U)) is finite but $U \subseteq$ Int (Cl (U)), therefore U is finite i.e. x is an isolated point. #

Theorem 3.1.15 $C_F(X) \subseteq C_K(X)$, and the inclusion may be proper.

Let $f \in C_F(X)$. Then supp f = coz f is finite, so supp f is compact.

Now, let $X = [0, 1]$ be a subspace of $\mathbb{R}$ with usual topology. Then X is compact so $C_K(X) = C(X)$, hence $1 \in C_K(X)$, but $1 \notin C_F(X)$ since coz $1 = X$, which is compact infinite set. #

Theorem 3.1.16 Let X be a pseudo-finite space. Then the ideal $C_F(X)$ coincides with $C_K(X)$.

Proof: Let $f \in C_K(X)$. Then supp f is compact, but X is pseudo-finite space so supp f is finite. Since coz $f \subseteq$ supp f, hence coz f is finite i.e. $f \in C_F(X)$.

Thus $C_K(X) \subseteq C_F(X)$ and by 3.1.15, $C_F(X) \subseteq C_K(X)$. So $C_F(X) = C_K(X)$. #

Theorem 3.1.17 Let X be a topological space. Then

1- $C_F(X) = C_K(X)$ if and only if X is a pseudo-discrete space.

2- $C_\infty(X) = C_F(X)$ if and only if X is a pseudo-discrete space with only a finite number of isolated points.

Proof:

1- ($\longleftarrow$) Shown in 3.1.16.

($\longrightarrow$) Suppose that $C_F(X) = C_K(X)$ and A is compact subset of X, suppose Int A is not finite and assume that $\{x_1, x_2, \ldots\}$ is an infinite subset of Int A.

Now, for each positive integer n, we define $f_n \in C(X)$ such that $f_n(x_n) = 1$ and $f_n(X \setminus \text{Int } A) = 0$. Then $f = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{f_n^{\,2}}{1+f_n^{\,2}}$ is clearly continuous and $f[X \setminus \text{Int } A] = 0$ and for every positive integer n, $f(x_n) \neq 0$. Since $\text{Cl}(X \setminus Z(f)) \subseteq \text{Cl}(\text{Int } A) \subseteq A$, so $\text{Cl}(X \setminus Z(f))$ is compact i.e. $f \in C_K(X)$, but $C_F(X) = C_K(X)$, therefore $X \setminus Z(f)$ is finite, which is a contradiction.

(2) Let $C_F(X) = C_\infty(X)$. Then $C_F(X) = C_K(X)$ and by part (1), X must be a pseudo-discrete space.

Now, suppose that the set of isolated points of X is an infinite subset.

Let $\{x_1, x_2, \ldots\}$ be a countable set of isolated points. Then define

$f_n \in C(X)$ to be such that $f_n(x_n) = \frac{1}{n}$ and $f_n[X \setminus \{x_n\}] = 0$. Clearly $f_n \in C(X)$.

Let $f = \sum_{n=1}^{\infty} \frac{f_n}{2^n}$. Then $f \in C(X)$, $f(x_n) = \frac{1}{n\ 2^n}$ and

$f[X \setminus \{x_1, x_2, \ldots, x_n, \ldots\}] = 0$, since $f(x_n) = \frac{1}{n\ 2^n} < \frac{1}{n}$, and for every m, we have

$\{x \in X: |f(x)| \geq \frac{1}{m}\} \subseteq \{x_1, x_2, \ldots, x_m\}$ i.e. for each m,

$\{x \in X: |f(x)| \geq \frac{1}{m}\}$ is compact and hence $f \in C_\infty(X)$.

But $X \setminus Z(f) = \{x_1, x_2, \ldots\}$ which is implies that $f \notin C_F(X)$, a contradiction.

$(\longleftarrow)$ Let X be a pseudo-discrete space with only a finite number of isolated points and $f \in C_\infty(X)$, but $f \notin C_F(X)$. Then since

$A_n = \{x \in X: |f(x)| > \frac{1}{n}\} \subseteq \{x \in X: |f(x)| \geq \frac{1}{n}\}$ and $\{x \in X: |f(x)| \geq \frac{1}{n}\}$ is compact, so its interior is finite since X is pseudo-discrete space.

Hence A_n is a finite open set for every positive integer n. On other hand $f \notin C_F(X)$, i.e. there is an infinite set $\{x_1, x_2, \ldots\}$ such that for each n, $f(x_n) \neq 0$. But for every n, there is an integer k_n such that $|f(x_n)| > \frac{1}{k_n}$, so $x_n \in A_{K_n}$ since A_{K_n} is open set, x_n is an isolated point, a contradiction. #

Corollary 3.1.18 If X is pseudo-discrete space with only finite number of isolated points, then $C_F(X) = C_\infty(X) = C_K(X)$.

Example 3.1.19 $C_F(\mathbb{Q}) = C_\infty(\mathbb{Q}) = C_K(\mathbb{Q}) = \{0\}$, but $C_F(\mathbb{N}) = C_K(\mathbb{N}) \subset C_\infty(\mathbb{N})$, since $\mathbb{N}$ has infinitely many isolated points

Proposition 3.1.20

Let $C_{\iota\sigma}(X) = \{f \in C(X): X \setminus Z(f)$ is locally compact σ-compact$\}$.

Then $C_{\iota\sigma}(X)$ is the smallest z-ideal in C(X) containing $C_\infty(X)$ or $C_{\iota\sigma}(X)$ is all of C(X).

Proof: Let $h \in C(X)$ and $f, g \in C_{\iota\sigma}(X)$.

Then $X \setminus Z(hf) \subseteq X \setminus Z(f)$ and $X \setminus Z(f-g) \subseteq (X \setminus Z(f)) \cup (X \setminus Z(g))$, so $X \setminus Z(hf)$ and $X \setminus Z(f-g)$ are locally compact $\sigma-compact$.

Hence $hf \in C_{\iota\sigma}(X)$ and $f-g \in C_{\iota\sigma}(X)$. Thus $C_{\iota\sigma}(X)$ is an ideal in C(X).

Let $g \in C(X)$ such that $Z(g) = Z(f)$ for some $f \in C_{\iota\sigma}(X)$. Then $X \setminus Z(f) = X \setminus Z(g)$,

so g$\in C_{\iota\sigma}(X)$. Hence $C_{\iota\sigma}(X)$ is a z-ideal .

Now, suppose that I is a z-ideal in C(X) such that $C_{\infty}(X) \subseteq I$. *If* $f \in C_{\iota\sigma}(X)$, then

X \ Z (*f*) is locally compact $\sigma-compact$ and hence by 2.1.13, there exists

$g \in C_{\infty}(X)$ such that Z (*f*) = Z (*g*).

But $g \in C_{\infty}(X) \subseteq I$ and I is a z-ideal, hence $f \in I$.

Thus $C_{\iota\sigma}(X) \subseteq I$, and clearly $C_{\infty}(X) \subseteq C_{\iota\sigma}(X)$. #

From 3.1.20, we note that $C_{\iota\sigma}(X)$ = C(X) if and only if X is a locally compact

$\sigma-compact$ space.

Chapter 4

Ideals in $C_\infty(X)$

In this chapter, we study the structure of the prime and maximal ideals in $C_\infty(X)$, we also show that some proper ideals in $C_\infty(X)$ may not be contained in a maximal ideal. We study essential and uniform ideals. We give examples and counter examples when necessary.

The material of this chapter can be found in (Aliabad, et al., 2004).

4.1 Ideals in $C_\infty(X)$.

Definition 4.1.1 Let X be a topological space. Then Spec(C_∞(X)) is the set of all prime ideals in $C_\infty(X)$.

Proposition 4.1.2 Let X be a completely regular Hausdroff space. Then

$Spec\ (C_\infty(X)) = \{ P^* \cap C_\infty(X) : P^* is$ a prime ideal in $C^*(X)$ and $C_\infty(X) \not\subset P^*\}$.

Proof :

Let P^* be a prime ideal in $C^*(X)$ with $C_\infty(X) \not\subset P^*$. Then clearly $P^* \cap C_\infty(X)$ is prime ideal in $C_\infty(X)$.

Conversely, let P_∞ be a prime ideal in $C_\infty(X)$. Then P_∞ is an ideal in $C^*(X)$, since if $f, h \in P_\infty$ and $g \in C^*(X)$,

then clearly $f - h \in P_\infty$ and $fg = f^{\frac{1}{3}} f^{\frac{2}{3}} g$ but $f^{\frac{2}{3}} g \in C_\infty(X)$, since it is an ideal in $C^*(X)$ and $f^{\frac{1}{3}}, f^{\frac{2}{3}} \in C_\infty(X)$ and $f^{\frac{1}{3}} \in P_\infty$, because $f^{\frac{1}{3}} f^{\frac{2}{3}} = f \in P_\infty$ which is prime, hence $fg \in P_\infty$.#

Lemma 4.1.3 Let I be an ideal in a commutative ring R. Suppose Q and P are ideals in R and P is prime. If P does not contain I and $Q \cap I \subseteq P \cap I$, then $Q \subseteq P$. In particular, if Q is a prime ideal and $Q \cap I = P \cap I$, then $Q = P$.

Proof:

Let $Q \cap I \subseteq P \cap I$. Then $Q \cap I \subseteq P$. Since P is prime and $I \not\subset P$, then $Q \subseteq P$. #

Proposition 4.1.4 Let D be the collection of all prime ideals in $C^*(X)$ which do not contain $C_\infty(X)$. Then $\varphi : D \longrightarrow Spec(C_\infty(X))$ defined by $\varphi(P^*) = P^* \cap C_\infty(X)$ is a one to-one correspondence.

Proof: Using 4.1.1, $P^* \cap \mathrm{C}_\infty(X) \in \mathrm{Spec}(\mathrm{C}_\infty(X))$, and also clearly φ is well-defined.

Now, suppose $P_1^*, P_2^* \in D$.

Let $\varphi(P_1^*) = \varphi(P_2^*)$. Then $P_1^* \cap \mathrm{C}_\infty(X) = P_2^* \cap \mathrm{C}_\infty(X)$.

Hence $P_1^* = P_2^*$, by 4.1.3. Thus φ is one to-one.

Let $P^* \cap C_\infty(X) \in spec(C_\infty(X))$, where P^* is a prime ideal in $C^*(X)$ and $C_\infty(X) \not\subset P^*$. Then $\varphi(P^*) = P^* \cap C_\infty(X)$.

Hence φ is one to-one corresponding. #

Proposition 4.1.4 shows that every prime ideal P_∞ of $C_\infty(X)$ has a unique representation of the form $P_\infty = P^* \cap \mathrm{C}_\infty(X)$, where P^* is a prime ideal in $C^*(X)$.

Proposition 4.1.5 For every Hausdroff space X, whenever $C_\infty(X) \neq (0)$, then there exists a locally compact space Y such that $C_\infty(X) \cong \mathrm{C}_\infty(Y)$. In fact the space Y may be considered as a nonempty open locally compact subspace of X.

Proof: Let Y be the set of all points in X which has compact neighborhoods. Then Y is a locally compact open subspace of X, and since $C_\infty(X) \neq \{0\}$, $Y \neq \varnothing$.

We may also assume that $Y \neq X$, for otherwise X itself would be a locally compact space.

Define $\sigma : C_\infty(X) \longrightarrow C_\infty(Y)$ by $\sigma(f) = f\ |_Y$, $\forall f \in C_\infty(X)$.

Now, σ is one to one function since by 2.1.14, $f(X \setminus Y) = 0$,

and also σ is onto since

if $g \in C_\infty(Y)$, then we define $g^* : X \longrightarrow \mathbb{R}$

such that $g^*(x) = g(x)$, $\forall x \in Y$ and $g^*(x) = 0$, $\forall x \in X \setminus Y$.

To see the continuity of g^*, it is enough to show that g^* is continuous on the non-empty set X \ Y. Let $x \in X \setminus Y$ and $\varepsilon > 0$. Then the set $\{x \in Y : |g(x)| \geq \varepsilon\}$ is compact in Y and hence compact in X.

Therefore $G = X \setminus \{x \in Y : |g(x)| \geq \varepsilon\} = \{x \in X : |g^*(x)| < \varepsilon\}$ is an open set in X and $g^*(G) \subseteq (-\varepsilon, \varepsilon)$. So g^* is continuous at $x \in X \setminus Y$. On the other hand,

$\{x \in X : |g^*(x)| \geq \frac{1}{n}\} = \{x \in Y : |g(x)| \geq \frac{1}{n}\}$ which implies that $g^* \in C_\infty(X)$.

Now $\sigma(g^*) = g$. Thus σ is onto.

Finally, let $f, g \in C_\infty(X)$. Then clearly

$\sigma(f + g) = \sigma(f) + \sigma(g)$ and $\sigma(fg) = \sigma(f)\ \sigma(g)$. So σ is an isomorphism of rings.

Thus $C_\infty(X) \cong C_\infty(Y)$. #

Theorem 4.1.7 Let X be a topological space. Then $C_\infty(X) \neq (0)$ if and only if $Spec(C_\infty(X)) \neq \varnothing$.

Proof: Assume $C_\infty(X) \neq (0)$. Then by 4.1.5, there exists a nonempty locally compact space Y such that $C_\infty(X) \cong C_\infty(Y)$. Hence it is enough to show that $Spec(C_\infty(Y)) \neq \varnothing$.

Now if Y is compact, then $C_\infty(X) = C^*(X)$ and clearly $Spec(C_\infty(X)) \neq \emptyset$.

So suppose Y is not compact, and let P^* be a fixed prime ideal in $C^*(Y)$.

Therefore $C_\infty(Y)$ is free by 2.1.10, and hence no fixed prime ideal of $C^*(Y)$ contains $C_\infty(Y)$ and so $C_\infty(Y) \not\subset P^*$.

On the other hand, $C_\infty(Y)$ is a free ideal in $C^*(Y)$, so by 2.1.21, $C_\infty(Y)$ is an essential ideal in $C^*(Y)$. Hence $C_\infty(Y)$ intersects every non-zero ideal in $C^*(Y)$ non-trivially.

Thus $P^* \cap C_\infty(Y) \neq (0)$, and we have $C_\infty(Y) \not\subset P^*$. So $Spec(C_\infty(Y)) \neq \emptyset$.

Conversely, if $C_\infty(X) = (0)$, then $Spec(C_\infty(X)) = \emptyset$. #

Proposition 4.1.8 A locally compact Hausdroff space X is $\infty-compact$ if and only if every prime ideal in $C_\infty(X)$ is fixed.

Proof: Suppose X is an $\infty-compact$ space.

Let P_∞ be a prime ideal in $C_\infty(X)$. Then by 4.1.1, there exists a prime ideal P^* in $C^*(X)$ such that $P_\infty = P^* \cap C_\infty(X)$, where $C_\infty(X) \not\subset P^*$.

Now, if P^* is a free ideal, then $C_K(X) \subseteq P^*$, since $C_K(X)$ is the intersection of all free ideals in C(X). But $C_\infty(X) = C_K(X)$, so $C_\infty(X) \subseteq P^*$ which is a contradiction.

Hence P^* is a fixed ideal and therefore P_∞ is fixed.

Conversely, suppose that every prime ideal in $C_\infty(X)$ is fixed but X is not $\infty-compact$. Hence there exists $f \in C_\infty(X)$ such that $f \notin C_K(X)$. Now consider the prime ideal P^* in $C^*(X)$ containing $C_K(X)$ but not *f*.

Therefore $C_K(X) \subseteq P^* \cap C_\infty(X) = P_\infty$, and P_∞ is a prime ideal in $C_\infty(X)$.

Now, since X is a locally compact non-compact, so $C_K(X)$ is free and therefore P^* is free. Thus P_∞ is free, which is a contradiction. #

Theorem 4.1.9 Let X be a topological space. Then $C_\infty(X)$ is not contained in any prime ideal of C(X) if and only if X is locally compact $\sigma-compact$.

Proof: Let P be a prime ideal in C(X) such that $C_\infty(X) \subseteq P$. Then there exists a maximal ideal M in C(X) such that $C_\infty(X) \subseteq M$, since $C_\infty(X) \subseteq P$ and P is proper.

So $C_{\iota\sigma}(X) \subseteq M$, since $C_{\iota\sigma}(X)$ is the smallest z-ideal containing $C_\infty(X)$.

Hence $C_{\iota\sigma}(X)$ is a proper ideal in C(X) and so X is not locally compact or not $\sigma-compact$.

Conversely, assume X is not locally compact or not $\sigma-compact$. Then $C_{\iota\sigma}(X)$ is a proper ideal in C(X), so $C_{\iota\sigma}(X)$ is contained in a maximal ideal of C(X).

Hence $C_{\iota\sigma}(X)$ is contained in a prime ideal of C(X) which also contains $C_\infty(X)$. #

Note that since $\mathbb{R}$ is locally compact $\sigma-compact$ space, then it follows by 4.1.9 that $C_\infty(\mathbb{R})$ is not contained in any prime ideal of $C(\mathbb{R})$. On the other hand $\mathbb{Q}$ is $\sigma-compact$ but not locally compact and $C_\infty(\mathbb{Q}) = \{0\}$ is contained in each prime ideal of $C(\mathbb{Q})$.

Lemma 4.1.10 Let X be a topological space. Then $C_\infty(X)$ contains a prime ideal of $C^*(X)$ if and only if X is an almost compact space.

Proof: Let P^* be a prime ideal of $C^*(X)$ and $P^* \subseteq C_\infty(X)$. Then

$P^* \subseteq \bigcap_{x \in \beta X - X} M^{*x}$ and since every prime ideal of $C^*(X)$ is contained in a unique maximal ideal in $C^*(X)$,

hence $C_\infty(X) = M^{*x}$, where $\beta X - X = \{x\}$. Thus X is an almost compact space.

Conversely, let X be an almost compact space. Then $C_\infty(X) = M^{*x}$, but every maximal ideal in $C^*(X)$ is a prime ideal in $C^*(X)$, so $C_\infty(X)$ contains a prime ideal of $C^*(X)$.#

Theorem 4.1.11 Let A be a commutative algebra over the rationals with unity. Let I be an ideal of A. Then an ideal D of I is maximal ideal of I if and only if D = $M \cap I$ for some maximal ideal M in A.

Proof: See (Rudd, 1971)

Theorem 4.1.12 Every maximal ideal of $C_\infty(X)$ is of the form $M_x \cap C_\infty(X)$, where M_x is a fixed maximal ideal in C(X) and the point *x* has a compact neighborhood.

Proof: Since $C_\infty(X)$ is the intersection of all free maximal ideals in $C^*(X)$, see 2.1.7. So every maximal ideal of $C_\infty(X)$ is of the form $M_p^* \cap C_\infty(X)$ where $p \in X$ and $C_\infty(X) \not\subseteq M_p^*$. But if $C_\infty(X) \subseteq M_p^*$ for some $p \in X$, then $f(p) = 0$ for all $f \in C_\infty(X)$ and by 2.1.4, the point *p* has no compact neighborhood. Hence if we consider *A* to be the set of all points of *X* which has no compact neighborhoods, then the collection of all maximal ideals of $C_\infty(X)$ is $\{M_x^* \cap C_\infty(X) : x \in (X \setminus A) = X_L\}$. On the other hand, $M_x^* = M_x \cap C^*(X)$ for all $x \in X$ implies that every maximal ideal of $C_\infty(X)$ is of the form $M_x \cap C_\infty(X)$, where $x \in X \setminus A$.

If $p \in \beta X - X$, then $C_\infty(X) \subseteq M^P \cap C^*(X)$.

So $M^{P^*} \cap C_\infty(X) = C_\infty(X)$, which is not a maximal ideal in $C_\infty(X)$.

Note that if X is locally compact, then the only maximal ideals of $C_\infty(X)$ are of the form $M_p \cap C_\infty(X)$, where $p \in X$, (i.e. we have a one to-one corresponding between X

and the set of all maximal ideals of $C_\infty(X)$), since if $p \in \beta X - X$, then

$C_\infty(X) \subseteq M^{*P} \subseteq M^P$ and so $M^P \cap C_\infty(X) = C_\infty(X)$ is not maximal in $C_\infty(X)$.

If $p \in X$, then $M^P = M_P$ is a fixed ideal and so $C_\infty(X) \not\subseteq M_P$, since otherwise

$C_\infty(X)$ would be fixed which is not true, since X is locally compact non-compact space

and so $C_\infty(X)$ is free. #

Examples 4.1.13

1- The space $\mathbb{R}$ is locally compact, so it follows by 4.1.12 that

$Max(C_\infty(\mathbb{R})) = \{M_x \cap C_\infty(\mathbb{R}) : x \in \mathbb{R}\}$. Note that in this case $C_K(\mathbb{R})$ is a proper ideal

of $C_\infty(\mathbb{R})$ which is not contained in any maximal ideal, because $C_K(\mathbb{R})$ is free while

all maximal ideals of $C_\infty(\mathbb{R})$ are fixed.

2- Let $X = [0,1]$ with all points are isolated except for $x = 0$ has its usual neighbors in

the usual topology. Then $X_L = (0,1]$, $C_\infty(X) \subset M_0$, and

$Max(C_\infty(X)) = \{M_x \cap C_\infty(X) : x \in (0,1]\}$.

Theorem 4.1.14 Two locally compact spaces X and Y are homeomorphic if and only if

$C_\infty(X) \cong C_\infty(Y)$.

Proof: If $X \approx Y$, then clearly $C_\infty(X) \cong C_\infty(Y)$.

If $C_\infty(X) \cong C_\infty(Y)$, then there is a 1-1 correspondence between the maximal ideals of

$C_\infty(X)$ and $C_\infty(Y)$ and so a 1-1 correspondence between X and Y. #

Definition 4.1.15 The set *E* is called nowhere compact if *E* does not contain any

nonempty compact neighborhood.

Definition 4.1.16 A non-zero ideal E in a ring *R* is called uniform ideal if any two non-

zero ideals contained in E intersect none trivially.

Proposition 4.1.17 Let X be a topological space. Then

(a) An ideal E in $C_\infty(X)$ is essential if and only if $\bigcap Z[E]$ is nowhere compact.

(b) An ideal I in $C_\infty(X)$ is uniform if and only if

$I=(f)$ for some $f\in C_\infty(X)$, where $X\setminus Z(f)$ is a singleton.

Proof: (a) Let E be an essential ideal in $C_\infty(X)$, and let B = $\bigcap Z[E]$.

Assume that there exists a compact set A with $A\subseteq B$ and int $A\neq\varnothing$.

Suppose $a\in$ int A and define $f\in C(X)$ such that $f(X\setminus \text{int } A)=\{0\}$ and $f(a)=1$.

Hence $\{x\in X:|f(x)|\geq \frac{1}{n}\}\subseteq A$ and so $\{x\in X:|f(x)|\geq \frac{1}{n}\}$ is compact

(i.e. $f\in C_\infty(X)$).

Now, if there exists $g\in C_\infty(X)$ such that $g\in(f)\cap E$, then $Z(f)\subseteq Z(g)$ implies

that $X\setminus Z(g)\subseteq X\setminus Z(f)\subseteq A\subseteq B\subseteq Z(g)$.

Hence g = 0 which contradicts the essentiality of E in $C_\infty(X)$.

Conversely, let $\bigcap Z[E]$ be nowhere compact, $0\neq f\in C_\infty(X)$ and $a\in X\setminus Z(f)$.

Then there exists $n\in\mathbb{N}$ such that $|f(x)|\geq \frac{1}{n}$ and hence a is in the compact set

$\{x\in X:|f(x)|\geq \frac{1}{n}\}$.

Now, since $\bigcap Z[E]$ is nowhere compact, so there exists

$b\in\{x\in X:|f(x)|\geq \frac{1}{n}\}\setminus\bigcap Z(E)$. Therefore there exists $g\in E$ such that $g(b)\neq 0$.

Hence $0\neq fg\in(f)\cap E$. Thus E is essential in $C_\infty(X)$.

(b) Suppose I is a uniform ideal in $C_\infty(X)$ and $f\in I$.

First we show that $X\setminus Z(f)$ is a singleton.

Let $x_0, y_0 \in X \setminus Z(f)$ and $x_0 \neq y_0$. Then by 2.1.13, $X \setminus Z(f)$ is a locally compact subspace of X and hence there exists two disjoint compact neighborhoods G and H in $X \setminus Z(f)$ of x_0 and y_0 respectively.

Now since $X \setminus Z(f)$ is open in X, then G and H are also compact neighborhoods in X.

Define two functions $g,h \in C(X)$ such that

$g(x_0) = 1 = h(y_0)$ and $g(X \setminus \operatorname{int} G) = \{0\} = h(X \setminus \operatorname{int} H)$.

Hence $g, h \in C_\infty(X)$ since $\{x \in X : |g(x)| \geq \frac{1}{n}\} \subseteq G$ and $\{x \in X : |h(x)| \geq \frac{1}{n}\} \subseteq H$.

Now, consider the principal ideals (fg) and (fh) of I. Then there exists

$0 \neq k \in (fg) \cap (fh)$ since I is a uniform ideal.

Hence there exists $z \in X \setminus Z(g)$ with $k(z) \neq 0$, but $kg = 0$ which contradicts

$k(z)g(z) \neq 0$.

Thus $X \setminus Z(f)$ is a singleton, say $X \setminus Z(f) = \{x_0\}$.

Next we show that for every $g \in I$, we have also $X \setminus Z(g) = \{x_0\}$.

Suppose $X \setminus Z(g) = \{y_0\}$ and $y_0 \neq x_0$.

Let (f) and (g) are principal subideals. Then $(f) \cap (g) = (0)$, since if $h \in (f) \cap (g)$, then $Z(f) \cup Z(g) = X \subseteq Z(h)$. Hence $h = 0$, which contradicts the uniformity of I.

Thus $X \setminus Z(g) = \{x_0\}$.

Therefore we have shown that there exists an isolated point $x_0 \in X$ such that

$X \setminus Z(g) = \{x_0\}, \forall f \in I$.

Finally, let $f, g \in I$ and $f(x_0) = \alpha$. Then there exists $n \in \mathbb{N}$ such that

$|\alpha| \geq \frac{1}{n}$, so $\{x \in X : |f(x)| < \frac{1}{n}\} \subseteq Z(g)$ since $X \setminus Z(g) = \{x_0\}$.

Hence g is a multiple of f, by 2.1.11. Thus $I = (f)$.

Conversely, Suppose L and J are non-zero ideals in I .

Let $f \in L \cap J$, $g \in L$ and $h \in J$. Then $Z(h) \cup Z(g) = X \setminus \{x_0\} \subseteq Z(f)$.

So $f \neq 0$. Hence I is a uniform. #

Examples 4.1.18

1) Since $\mathbb{R}$ is locally compact non-pseudocompact, it follows that $C_K(\mathbb{R})$ is a proper free ideal of $C_\infty(\mathbb{R})$, so $\bigcap Z[C_K(\mathbb{R})] = \varnothing$ is nowhere compact. Thus $C_K(\mathbb{R})$ is an essential ideal of $C_\infty(\mathbb{R})$.

2) Let $X = [0,1]$ with all points are isolated except for $x = 0$ has its usual neighbors in the usual topology. Define the function $f(x) = \begin{cases} 1 & x = \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$

Then $f \in C_\infty(X)$ and the ideal (f) is uniform in $C_\infty(X)$, since $X \setminus Z(f) = \{\frac{1}{2}\}$ is a singleton.

4.2 Further Properties of ideals in $C_\infty(X)$.

Definition 4.2.1 Let X be a topological space. Then X is called $P_\infty-space$ if $Z(f)$ is open for every $f \in C_\infty(X)$.

Example 4.2.2 Let S be a P-space. Let the space X be the disjoint union of the spaces S and $\mathbb{Q}$. Then by 2.1.14, for every $f \in C_\infty(X)$, we have $f(\mathbb{Q}) = \{0\}$. Now, $Z(f)$ is open for every $f \in C_\infty(X)$ since S is a P-space. Hence X is $P_\infty-space$, but $\mathbb{Q}$ is not a P-space. Thus X is not a P- space.

The above example shows that a $P_\infty-$ space need not be a P-space.

Proposition 4.2.3 If X is a P – space, and every function in C(X) is bounded on a subset S, then S is finite.

Proof: See (Gillman and Jerison, 1976). #

Theorem 4.2.2 Every countable subset of a P-space is discrete and closed.

Proof: Let Y = $\{x_1, x_2, ...\} \subseteq$ X.

Let $G_1 = X$

$G_2 = X - \{x_2\}$

$G_3 = X - \{x_2, x_3\}$

.. ..

$G_n = X - \{x_2, x_3, ..., x_n\}$.

Then G_i is open in X for all i.

Now, G = $\bigcap_{i=1}^{\infty} G_i$ is open in X, since it is a P-space.

$\{x_1\}$ = G $\cap$ Y is open in Y, similarly $\{x_2\}, \{x_3\}$, ... are open in Y.

Hence Y is discrete. Now, to show that Y is closed, let x$\in$X $-$Y

Let $H_1 = X - \{x_1\}$

$H_2 = X - \{x_1, x_2\}$

..

..

$H_n = X - \{x_1, x_2, ..., x_n\}$.

Then H_i is open in X for all i, H = $\bigcap_{i=1}^{\infty} H_i$ is G_δ- set in X and so is open.

But x $\in H \subseteq X - Y$, i.e. $X - Y$ is open and Y is closed. #

Corollary 4.2.5 If *X* is a compact P-space, then *X* is finite.

Proof: If *X* is an infinite set, then it contains an infinite countable subspace *Y*, so by 4.2.4 *Y* is closed and discrete. Hence *Y* is compact. But a compact discrete space must be finite, contradiction. Hence *X* is finite. #

Proposition 4.2.6 Let *X* be a locally compact Hausdroff space. If X is a $P_\infty - space$, then it is also a P-space.

Proof: Assume *X* is a $P_\infty - space$. Then $M_x^* \cap C_\infty(X) = O_x^* \cap C_\infty(X)$, $\forall$ x $\in$X.

Since M_x^* is prime in $C^*(X)$, so $M_x^* = O_x^*$ or $C_\infty(X) \subseteq O_x^*$ by 4.1.3.

Now, let *K* and *H* be compact neighborhoods of x such that $K \subseteq$ Int *H*. Then define $g \in$C(X) with *g* (*K*) = {1} and *g*(*X* \ Int *H*) = {0}, by Urysohn's lemma.

So $X \setminus Z(g) \subseteq H$, hence $g \in C_K(X) \subseteq C_\infty(X)$, but $g \notin O_x^*$. So $C_\infty(X) \not\subset O_x^*$.

Hence $M_x^* = O_x^*$, $\forall$ x $\in$X and so $M_x = O_x$, $\forall x \in X$. Therefore *X* is a P-space. #

Theorem 4.2.7 For a topological space X. the following statements are equivalent:

1) $C_\infty(X)$ is a regular ring.

2) Every open locally compact σ-compact set in *X* is compact.

3) For all $f \in C_\infty(X)$, $X \setminus Z(f)$ is compact.

4) *X* is an ∞-compact $P_\infty - space$.

5) For all $p \in X$, $M_P \cap C_\infty(X) = O_P \cap C_K(X)$.

Proof: $(1) \longrightarrow (2)$

Assume $C_\infty(X)$ is a regular ring.

Using 2.1.14, we get that every open locally compact σ-compact set is of the form $X \setminus Z(f)$ for some $f \in C_\infty(X)$. Then there exists $g \in C_\infty(X)$ such that $f^2 g = f$.

So $f(fg-1) = 0$ implies that $\{x : (fg)(x) \neq 1\} = Z(f)$, i.e. $Z(f)$ is open. On the other hand $g(x) = \dfrac{1}{f(x)}$ for every $x \in X \setminus Z(f)$ and hence $g(x) > \dfrac{1}{N}$, where N is an upper bound for $|f|$ since every function of $C_\infty(X)$ is bounded, by 2.1.5.

Therefore $(X \setminus Z(f) \subseteq \{x \in X : |g(x)| \geq \dfrac{1}{N}\} = A_N$.

Since $X \setminus Z(f)$ is closed and A_N is compact, then $X \setminus Z(f)$ is compact.

$(2) \longrightarrow (3) \longrightarrow (4) \longrightarrow (5)$ Evident.

$(5) \longrightarrow (1)$

Assume that $\forall$ $p \in X$, $M_P \cap C_\infty(X) = O_P \cap C_K(X)$. Then for every $f \in C_\infty(X)$, $Z(f)$ is open, by 2.2.14,and $X \setminus Z(f)$ is compact. Hence $X \setminus Z(f)$ is finite, by 4.2.5.

Now, for every $f \in C_\infty(X)$, we define $g(x) = \begin{cases} 0 & x \in Z(f) \\ \dfrac{1}{f(x)} & x \in X / Z(f) \end{cases}$.

Hence $g \in C(X)$ and $\{x \in X : |g(x)| \geq \dfrac{1}{n}\} \subseteq X \setminus Z(f)$, so $\{x \in X : |g(x)| \geq \dfrac{1}{n}\}$ is compact, since $X \setminus Z(f)$ is compact.

So $g \in C_\infty(X)$ and $f^2 g = f$. Thus $C_\infty(X)$ is a regular ring. #

Theorem 4.2.8 let X be locally compact Hausdroff space. Then $C_\infty(X)$ is regular ring if and only if X is finite.

Proof: $\longrightarrow$

Let X be finite, $f \in C_\infty(X)\setminus\{0\}$ and $g(x) = \begin{cases} \dfrac{1}{f(x)} & x \in cozf \\ 0 & x \in Z(f) \end{cases}$.

Then $g \in$ C(X) and for each $n \in \mathbb{N}$ $\{ x \in X : |g(x)| \geq \frac{1}{n}\}$ is finite and so it is compact.

Thus; $g \in C_\infty(X)$ and $f = f^2 g$. Hence $C_\infty(X)$ is a regular ring.

$\longleftarrow$

Let $C_\infty(X)$ is regular ring. Then by 4.2.7, *X* is an ∞-compact $P_\infty - space$.

Hence by 4.2.6, X is P-space and so pseudocompact space, by 3.1.4. Thus X is finite.

#

Proposition 4.2.9 If *R* is regular ring, then it is a zero dimensional ring.

Proof: Suppose *R* is a regular ring.

Let *P* be a prime ideal in *R*, and *J* be any ideal in *R* such that $P \subsetneq J$. Then there exists an element $x \in J$ and $x \notin P$.

Since *R* is a regular $x = x^2 y$ for some y $\in$ R. So $x - x^2 y = 0 \in P$ thus $x(1-xy) \in P$, but $x \notin P$ and *P* is a prime ideal in *R*.

Hence $1-xy \in P$, and so $1-xy \in J$ since $P \subsetneq J$.

Therefore $1-xy+xy = 1 \in J$ because x $\in$ J and xy $\in$ J.

Hence *J* = *R* which implies *P* is a maximal ideal in R.

Thus *R* is a zero dimensional ring. #

Definition 4.2.10 A ring *R* has a finite Goldie dimension if there exists an integer $n > 0$ such that any direct sum of non-zero ideals in *R* has always *m* terms, where $m \leq n$ and there is a direct sum of uniform ideals with *n* terms which is essential in *R*.

Proposition 4.2.11 $C_\infty(X)$ has a finite Goldie dimension if every open locally compact set in *X* is finite.

Proof: If $C_\infty(X) = (0)$, then every locally compact set in *X* is empty by 2.1.14.

Now suppose $C_\infty(X) \neq (0)$ has a finite Goldie dimension.

Let *G* be a locally compact open set in *X*. Then there exists $n > 0$ such that the direct sum of n uniform ideals $I_1, I_2, ..., I_n$ in $C_\infty(X)$ is an essential ideal *E* in $C_\infty(X)$.

Hence by 4.1.16, there is an isolated point $x_i \in X$ and $f_i \in I_i$ such that $I_i = (f_i)$, where

$X \setminus Z(f_i) = \{x_i\}$, for $i = 1, 2, ..., n$, and so $\bigcap Z[I] = X \setminus \{x_1, x_2, ..., x_n\}$.

But $X \setminus \{x_1, x_2, ..., x_n\}$ does not contain any non-empty compact neighborhood.

Thus $G \bigcap (X \setminus \{x_1, x_2, ..., x_n\}) = \varnothing$ and therefore $G \subseteq \{x_1, x_2, ..., x_n\}$, i.e. G is finite. #

Corollary 4.2.12 If *X* is a locally compact Hausdroff space, then $C_\infty(X)$ has a finite Goldie dimension if and only if *X* is finite.

Definition 4.2.13 A space *X* is called basically disconnected if for each $f \in C(X)$, supp f is clopen in *X*.

Definition 4.2.14 A commutative ring *R* is a p.p. (Bear) ring if for any $a \in R$ $(S \subseteq R)$, Ann(a) (Ann(S)) is generated by an idempotent.

Proposition 4.2.15 The space *X* is basically disconnected if and only if C(X) is a p.p. ring.

Proof: See (Brookshear, 1977). #

Theorem 4.2.16 Let *X* be a locally compact space. Then $C_\infty(X)$ is a p.p. ring if and only if *X* is basically disconnected compact space.

Proof: Let $C_\infty(X)$ be a p.p. ring. Then for every $0 \neq f \in C_\infty(X)$, there exists an idempotent $e \in C_\infty(X)$ such that Ann(*f*) = (e). Therefore $X \setminus Z(e) \subseteq Int\ Z(f)$.

Now, suppose $x \in Int\ Z(f)$ but $x \notin X \setminus Z(e)$.

Define $g \in C(X)$ such that $g(X \setminus Int\ K) = \{0\}$ and $g(x) = 1$, where K is a compact neighborhood of x contained in $Int\ (Z(f) \cap Z(e))$.

So $g \in C_K(X) \subseteq C_\infty(X)$ and $gf = 0$ but $g \notin (e)$, since e(x) = 0 and g(x) = 1, which is a contradiction. Hence $Int\ Z(f) \subseteq X \setminus Z(e)$, and so $Int\ Z(f) = X \setminus Z(e)$.

Thus $Cl(X \setminus Z(f)) = Z(e)$ is open, and hence *X* is basically disconnected.

Now, if we take $f \in C_K(X)$, then $Z(e)$ and $X \setminus Z(e)$ are compact and so *X* is compact.

Conversely, if *X* is compact and basically compact, then $C_\infty(X) = C(X)$.

So $C_\infty(X)$ is a p.p. ring, by 4.2.16. #

Corollary 4.2.17 Let *X* be a locally compact non-compact space. Then $C_\infty(X)$ is never a p.p. ring.

Theorem 4.2.18 A space *X* is basically disconnected if and only if βX is basically disconnected.

Proof: Assume *X* is a basically disconnected. Let $f \in C(\beta X)$ and $g = f|_X \in C(X)$.

Then supp f is open.

Define $h(x) = \begin{cases} 0 & x \in \text{supp } g \\ 1 & x \notin \text{supp } g \end{cases}$.

Then $h \in C(X)$. Moreover, X = supp $g \cup (X - \text{supp } g)$.

Now, $\beta X = Cl_{\beta X} \text{supp } g \cup Cl_{\beta X}(X - \text{supp } g) = \text{Cl}_{\beta X} \text{coz } f \cup Cl_{\beta X}(X - \text{supp } g)$.

Moreover, h^β is the continuous extension of *h* to βX with $h^\beta(x) = \begin{cases} 0 & x \in \text{supp } f \\ 1 & x \notin \text{supp } f \end{cases}$.

Hence supp f is open, and so βX is basically disconnected.

Conversely, assume βX is basically disconnected. Let $f \in C(X)$. Then

$$f^*(x)=\begin{cases}1 & f(x)\geq 1\\ f(x) & -1<f(x)<1.\\ -1 & f(x)\leq -1\end{cases}$$

Let f^{β} be the continuous extension of f^* to βX. Then supp f^{β} is open in βX.

Let $x \in \text{supp} f = \text{supp} f^* \subseteq \text{supp} f^{\beta}$ which is open. So, there exists U open in βX such that $x \in U \subseteq \text{supp} f^{\beta}$. Then $x \in U \cap X \subseteq \text{supp} f$, since if there exists $y \in U \cap X$ and $y \notin \text{supp} f$, then

$y \in (U \cap X) \cap Int_X\ Z(f) \subseteq Int_X\ Z(f)$ and $f \in O_y = O^y$.

Thus $y \in Int_{\beta X} \text{Cl}_{\beta X}\ Z(f) = Int_{\beta X} \text{Cl}_{\beta X}\ Z(f^*) = Int_{\beta X} Z(f^{\beta}) = \beta X - \text{supp} f^{\beta}$.

Contradiction, since $y \in U \subseteq \text{supp} f^{\beta}$. Therefore supp f is open and so X is basically disconnected. #

Example 4.2.19

1- $C_\infty(\mathbb{R})$ is not a p.p. ring since $\mathbb{R}$ is a locally compact non-compact space.

2- $C_\infty(\beta\mathbb{N})$ is a p.p. ring since $\beta\mathbb{N}$ is basically disconnected compact space.

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بعض خواص حلقة الاقترانات المتصلة المختفية في المالانهاية

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ملخص

ليكن X فضاءً منتظماً تماماً و هاوس دورف , و لتكن C(X) حلقة جميع الإقترانات الحقيقية المتصلة المعرفة على X . لتكن $C_\infty(X)$ مجموعة جميع الإقترانات المتصلة المختفية في المالانهاية. أي الاقترانات f في C(X) بحيث أن المجموعة $\{x \in X : |f(x)| \geq \frac{1}{n}\}$ متراصة لكل $n \in \mathbb{N}$.

في هذه الاطروحة درسنا الحلقة الجزئية $C_\infty(X)$, و أثبتنا أن $C_\infty(X)$ ليست بالضرورة مثالي في C(X). لقد أثبتنا أن $C_\infty(X)$ مثالي في C(X) إذا وفقط إذا كانت كل مجموعة مفتوحة ومتراصة محلياً محدودة إذا وفقط إذا كانت كل مجموعة مفتوحة ومتراصة محلياً و $\sigma-$ متراصة في X محدودة. لقد أثبتنا كذلك أنه إذا كان X متراصاً محلياً , فإن $C_\infty(X)$ مثالي في C(X) إذا وفقط إذا كان X متراصاً مظهرياً.

لقد شخصنا المثاليات الأولية و الأعظمية في $C_\infty(X)$. لقد برهنا أن المثاليات الأولية في $C_\infty(X)$ هي $\{P^* \cap C_\infty(X) : P^* \text{ is a prime ideal in } C^*(X) \text{ and } C_\infty(X) \not\subset P^*\}$.

لقد برهنا أن المثاليات الأعظمية في $C_\infty(X)$ هي $\{M^x \cap C_\infty(X) : x \text{ has compact nbhd}\}$.

لقد أثبتنا كذلك أن $C_\infty(X)$ قد تحتوي مثالي جزئي فعلي غير محتوى في مثالي أعظمي.